Tropical Polynomial Systems and Game Theory

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- Given a system of tropical polynomial equations or inequations, how to check the existence of, and compute solutions in $\mathbb{R}^n$.
- Main tools in the classical setting include the theory of resultants, Macaulay matrices and effective Null- and Positivstellensatz.
- In this thesis, we develop the tropical analog of the sparse Null- and Positivstellensatz, and use them to explore the solvability of tropical polynomial systems by means of mean payoff games, and nonlinear eigenvalues of (parametric) Shapley operators.

I - Tropical algebra and tropical polynomials

Tropical algebra and tropical polynomials

- **Tropical semiring** $\mathbb{T} := (\mathbb{R} \cup \{-\infty\}, \oplus, \odot, \mathbb{0}, \mathbb{1})$ with
  - addition $\oplus := \max$;
  - multiplication $\odot := +$;
  - zero element $\mathbb{0} := -\infty$;
  - unit element $\mathbb{1} := 0$.
- Invertible elements: $\mathbb{T}^* = \mathbb{R}$.
- Satisfies the usual properties of a field except **no additive inverse**.
- Tropical operations can be extended to vectors and matrices with coefficients in $\mathbb{T}$ to perform **tropical linear algebra**.

Tropical polynomials

- A **formal tropical polynomial** p in n variables is a map

$$\begin{array}{rcl} \mathbb{N}^n & \longrightarrow & \mathbb{T} \\ \alpha & \longmapsto & p_\alpha \end{array}$$

  such that $p_\alpha \neq \mathbb{0}$ for finitely many $\alpha \in \mathbb{N}^n$. We denote $p = \bigoplus_{\alpha \in \mathbb{N}^n} p_\alpha X^\alpha$.
- **Support** of p: $\mathrm{supp}(p) := \{\alpha \in \mathbb{N}^n : p_\alpha \neq \mathbb{0}\}$.
- **Polynomial function** associated to p:

$$\begin{array}{rcl} \mathbb{T}^n & \longrightarrow & \mathbb{T} \\ x & \longmapsto & p(x) := \bigoplus_{\alpha \in \mathcal{A}} p_\alpha \odot x^{\odot \alpha} \end{array}$$

  with $\mathcal{A} = \mathrm{supp}(p)$ and with the convention that $\mathbb{0}^{\odot 0} = \mathbb{1}$.

Tropical hypersurfaces

A point $x \in \mathbb{T}^n$ is a **root** of a polynomial p whenever the maximum in the expression

$$\bigoplus_{\alpha \in \mathcal{A}} p_\alpha \odot x^{\odot \alpha} = \max_{\alpha \in \mathcal{A}} (p_\alpha + \langle x, \alpha \rangle)$$

is attained for at least two distinct values of α (or is equal to $\mathbb{0}$). This is denoted as $p(x) \nabla \mathbb{0}$.

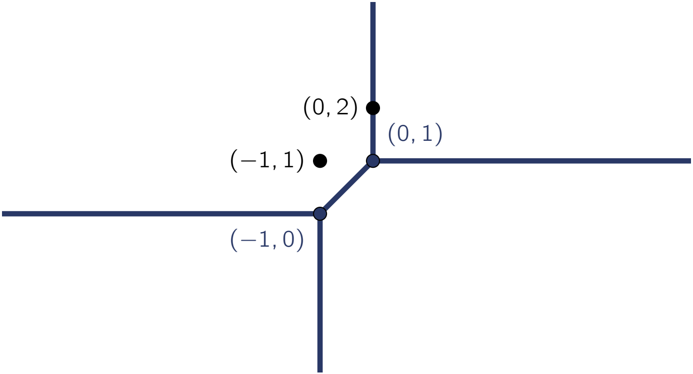
The **tropical hypersurface** associated to a tropical polynomial p is the set of its roots. It coincides with the non-differentiability locus of the fonction $x \mapsto p(x)$.

Kapranov's Theorem: If $\mathbb{K}$ is an algebraically closed non-Archimedean field with a nontrivial valuation, tropical hypersurfaces coincide with the closure of the image by the valuation map of classical hypersurfaces over $\mathbb{K}^*$.

Exemple : Let $f_1 = 1 \oplus 2x_1 \oplus 1x_2 \oplus 1x_1x_2$, then:

- $(0, 2)$ is a root of f_1 since the maximum of $f_1(0, 2) = 3$ is attained simultaneously by the monomials $1x_2$ and $1x_1x_2$;
- $(-1, 1)$ is not a root of f_1 since the maximum $f_1(-1, 1) = 2$ is attained only by the monomial $1x_2$.

The tropical hypersurface associated to the polynomial f_1 is:



Likewise, $y \in \mathbb{T}^m$ is said to be in the **tropical right null space** or **kernel** of a $\ell \times m$ matrix $A = (a_{ij})$ whenever for all $1 \leq i \leq \ell$, the maximum in the expression

$$\bigoplus_{j=1}^{m} a_{ij} \odot y_j = \max_{1 \leq j \leq m} (a_{ij} + y_j)$$

is achieved at least twice (or is equal to $\mathbb{0}$). This is also denoted as $A \odot y \; \nabla \; \mathbb{0}$.

List of the main contributions

- In chapter 2, we provide a tropical Nullstellensatz and Positivstellensatz adapted to sparse tropical polynomial systems and with an improved degree bound.
- In chapter 3, we propose a speed up of the classical value iteration algorithm for mean payoff games and use it to decide the solvability of a tropical polynomial system.
- In chapter 4, we present two algorithmic methods to certify the solvability of a tropical polynomial system as well as to compute its solution set.
- In chapter 5, we characterize the asymptotic behaviour of the Krasnoselskii-Mann iterates of nonexpansive fixed-point free polyhedral mappings.

II - The tropical Null- and Positivstellensatz

Position of the problem

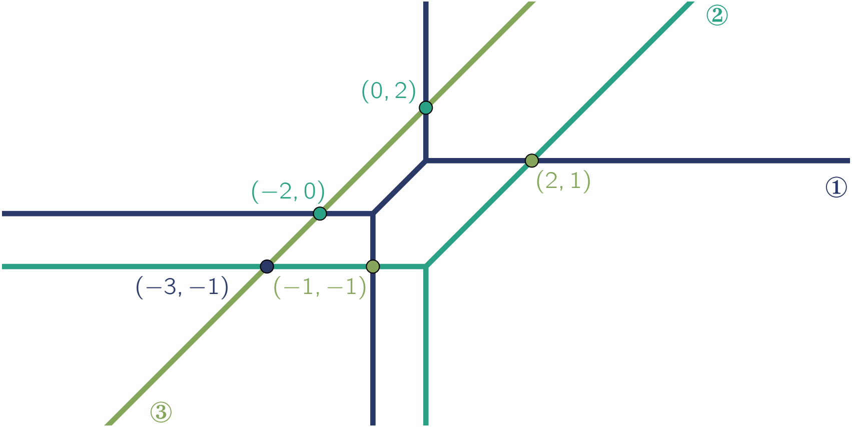
In the following, we fix a collection $f = (f_1, \dots, f_k)$ of k formal tropical polynomials in n variables, with respective supports $\mathcal{A} = (\mathcal{A}_1, \dots, \mathcal{A}_k)$ and degrees $(d_1, \dots, d_k)$.

Problem: Decide whether there is a common tropical zero $x \in (\mathbb{T}^*)^n$, that is such that $f_i(x) \mathrel{\nabla} \mathbb{0}$ for all $1 \leq i \leq n$.

Remark: The same question for a solution in $\mathbb{T}^n$ reduces to the $(\mathbb{T}^*)^n$ case by looking at all possible supports.

Figure: The arrangement of tropical varieties of the polynomials from the system

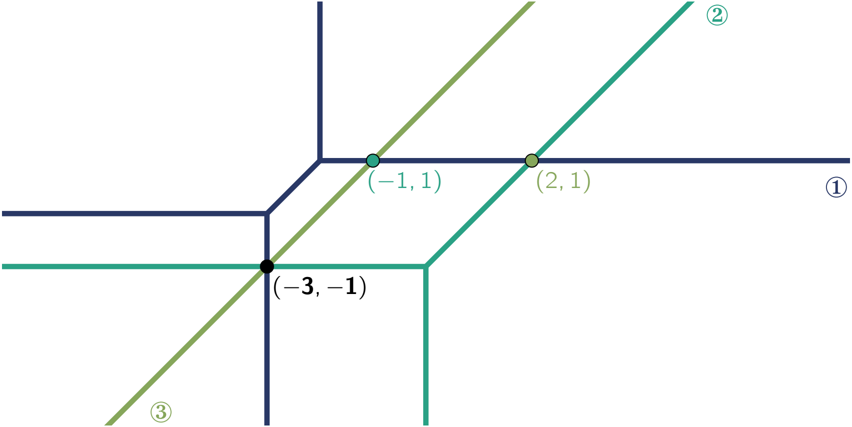
$$(E_1) : \begin{cases} f_1 &= 1 \oplus 2x_1 \oplus 1x_2 \oplus 1x_1x_2 \\ f_2 &= 0 \oplus 0x_1 \oplus 1x_2 \\ f_3 &= 2x_1 \oplus 0x_2 \ , \end{cases} .$$



Position of the problem

Figure: The arrangement of tropical varieties of the polynomials from the system

$$(E_2):\begin{cases} f_1 &= 1 \oplus 4x_1 \oplus 1x_2 \oplus 3x_1x_2 \\ f_2 &= 0 \oplus 0x_1 \oplus 1x_2 \\ f_3 &= 2x_1 \oplus 0x_2 \ , \end{cases} .$$



Position of the problem

Link with classical and tropical varieties:

- Polyhedral homotopy to solve polynomial systems (Huber, Sturmfels 1995; Leykin, Yu 2019)
- The Fundamental Theorem of Tropical Algebraic Geometry (Speyer, Sturmfels 2004; Payne 2013; see also Maclagan, Sturmfels 2015)
- Tropical primal and dual Nullstellensatz (Grigoriev 2012; Grigoriev, Podolskii 2018)
- Tropical resultant (Jensen, Yu 2013)
- Tropical homotopy continuation (Jensen 2016)

Varied applications:

- celestial mechanics (Hampton, Moeckel)
- max-out networks (Montúfar, Ren, Zhang)
- chemical reaction networks (Dickenstein, Feliu, Radulescu, Shiu)
- emergency call center (Allamigeon, Boyer, Gaubert)

The **tropical Macaulay matrix** (Grigoriev, 2012) associated to f is the infinite matrix $\mathcal{M} = (m_{(i,\alpha),\beta})$ indexed by $([n] \times \mathbb{Z}^n) \times \mathbb{Z}^n$, where $m_{(i,\alpha),\beta}$ corresponds to the coefficient of X^β in the polynomial $X^\alpha f_i$.

$$\mathcal{M} = \begin{array}{c} \\ f_1 \\ X_1 f_1 \\ \vdots \\ X^\alpha f_i \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & X_1 & \cdots & X^\beta & \cdots \end{array} \\ \begin{pmatrix} * & * & \cdots & * & \cdots \\ * & * & \cdots & * & \cdots \\ \vdots & \vdots & \ddots & \vdots & \\ * & * & \cdots & f_{i,\beta-\alpha} & \cdots \\ \vdots & \vdots & & \vdots & \ddots \end{pmatrix} \end{array}$$

One approch to the **Nullstellensatz** is a linearization method reducing the search of a solution of a polynomial system to the search of nonzero elements in the kernel of the Macaulay matrix.

- The **tropical Macaulay matrix** (Grigoriev, 2012) associated to f is the infinite matrix $\mathcal{M} = (m_{(i,\alpha),\beta})$ indexed by $([n] \times \mathbb{Z}^n) \times \mathbb{Z}^n$, where $m_{(i,\alpha),\beta}$ corresponds to the coefficient of X^β in the polynomial $X^\alpha f_i$.
- A finite subset $\mathcal{E}$ of $\mathbb{Z}^n$ yields a **finite** submatrix $\mathcal{M}_\mathcal{E}$ of $\mathcal{M}$ obtained by taking only the rows whose support is included in $\mathcal{E}$ and the columns indexed by $\mathcal{E}$.
- Set $\mathcal{M}_N := \mathcal{M}_\mathcal{E}$ for $\mathcal{E} = \{\alpha \in \mathbb{N}^n : \alpha_1 + \cdots + \alpha_n \leq N\}$.

The tropical Nullstellensatz

Conjecture [Grigoriev (2012)]: There exists an integer N such that

$$\exists x \in \mathbb{R}^n \text{ such that } f_i(x) \ \nabla \ \mathbb{0} \text{ for } i = 1, \dots, k$$

$$\iff$$

$$\exists y \in \mathbb{R}^m \text{ such that } \mathcal{M}_N \odot y \ \nabla \ \mathbb{0} \text{ with } m = \binom{N+n}{n} \ .$$

Answer:

- **Grigoriev, Podolskii (2018):** true for

$$N = (n+2)(d_1 + \cdots + d_k) \ .$$

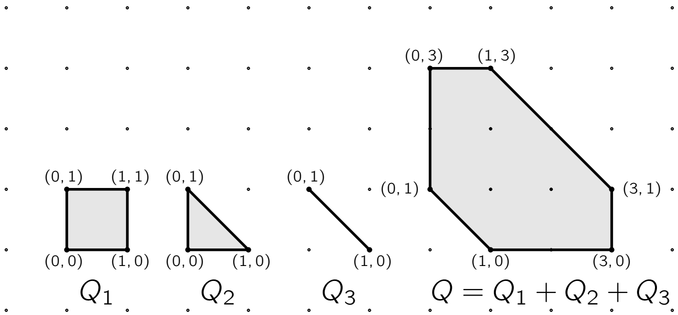
- **This thesis:** true for

$$N = d_1 + \cdots + d_k - 1$$

(and even $N = d_1 + \cdots + d_k - n$ in most cases) + adapted approch for the case of sparse polynomials.

- For $1 \le i \le k$, $Q_i := \mathrm{conv}(\mathcal{A}_i)$ is the **Newton polytope** of f_i.

Example: The Newton polytopes associated to both system (E_1) and system (E_2) and their Minkowski sum are as follow.



- **Canny-Emiris set** associated to f: $\mathcal{E} = (Q + \delta) \cap \mathbb{Z}^n$ with δ a generic vector in the linear space directing the affine hull of Q.

Initially used by Canny, Emiris (1993) and Sturmfels (1994) for the computation of the sparse resultant

Example: Considering again the systems (E_1) and (E_2), for

$$\delta = (-1 + \varepsilon, -1 + \varepsilon)$$

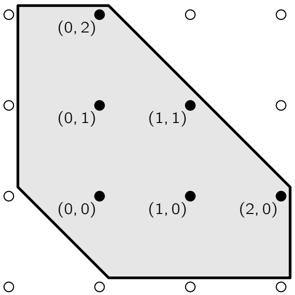
with $\varepsilon > 0$ sufficiently small, we obtain the Canny-Emiris set

$$\mathcal{E} := (Q + \delta) \cap \mathbb{Z}^n = \{(0,0), (1,0), (0,1), (2,0), (1,1), (0,2)\}$$

corresponding to the set of monomials $\{1, x_1, x_2, x_1^2, x_1 x_2, x_2^2\}$.

The tropical Nullstellensatz

Figure: The polytope $Q + \delta$ with $\delta = (-0.9, -0.9)$.



Nullstellensatz for Sparse Tropical Polynomial Systems

The system $f(x) \nabla \mathbb{0}$ has a solution $x \in \mathbb{R}^n$ iff there exists a vector $y \in \mathbb{R}^{\mathcal{E}'}$ in the tropical right null space of the submatrix $\mathcal{M}_{\mathcal{E}'}$ of $\mathcal{M}$, where $\mathcal{E}'$ is any subset of $\mathbb{Z}^n$ containing a nonempty Canny-Emiris set $\mathcal{E}$.

Corollary: *The system $f(x) \nabla \mathbb{0}$ has a solution $x \in \mathbb{R}^n$ if and only if the truncated Macaulay tropical linear system $\mathcal{M}_N \odot y \nabla \mathbb{0}$ has a solution $y \in \mathbb{R}^m$ for*

$$N = d_1 + \cdots + d_k - 1 \ ,$$

where $d_i = \deg(f_i)$ for all $1 \leq i \leq k$. Moreover, if Q has full dimension, then one can take $N = d_1 + \cdots + d_k - n$ in the previous statement.

Ingredients of the proof

A $d \times d$ tropical matrix $A = (a_{ij})_{1 \leq i,j \leq d}$ is **tropically diagonally dominant** whenever

$$a_{ii} > a_{ij}$$

for all $1 \leq i, j \leq d$ such that $i \neq j$.

Lemma: *If A is tropically diagonally dominant, then the only solution $y \in \mathbb{T}^d$ to the equation $A \odot y \;\nabla\; \mathbb{0}$ is $y = \mathbb{0}$.*

Ingredients of the proof

- If $f(x) \nabla \mathbb{0}$ has a solution $x \in \mathbb{R}^n$, then the **Veronese embedding** $y = \mathrm{ver}(x) := (x^p)_{p \in \mathcal{E}'}$ of x is a solution to $\mathcal{M}_{\mathcal{E}'} \odot y \nabla \mathbb{0}$.
- Otherwise we apply a construction from Canny and Emiris (1993) and Sturmfels (1994) but in a potentially **non generic** case to show that there is no finite vector $y \in \mathbb{R}^{\mathcal{E}'}$ in the tropical right null space of $\mathcal{M}_{\mathcal{E}'}$.

- The upper hull of the lifted support $\{(\alpha, f_{i,\alpha}) : \alpha \in \mathcal{A}_i\}$ is the graph of a function h_i with support Q_i.
- If $h := h_1 \,\square \cdots \square\, h_k$ where $\square$ denotes the sup-convolution, then $\mathrm{hypo}(h) = \mathrm{hypo}(h_1) + \cdots + \mathrm{hypo}(h_k)$ and moreover the supports of h is $Q = Q_1 + \cdots + Q_k$.
- The projection of $\mathrm{hypo}(h)$ onto Q yields a **coherent mixed subdivision** of Q.

Ingredients of the proof

- If $p \in \mathcal{E}$, then $(p - \delta, h(p - \delta))$ is in the **relative interior** of a facet F of hypo(h), and F can be written as $F_1 + \cdots + F_k$ with F_i faces of hypo(h_i).
- Since f does not have a common root, at least one F_i is a singleton. Consider the maximal index j such that $F_j = \{a_j\}$ is a **singleton**. The couple (j, a_j) is called the **row content** of p.
- If $p \in \mathcal{E}$ and if (j, a_j) is its row content, then the support of the polynomial $X^{p-a_j} f_j$ is included in $\mathcal{E}$. This allows us to construct a square submatrix $\mathcal{M}_{\mathcal{E}\mathcal{E}} = (m_{pp'})_{(p,p') \in \mathcal{E} \times \mathcal{E}}$ of $\mathcal{M}_{\mathcal{E}}$.

Ingredients of the proof

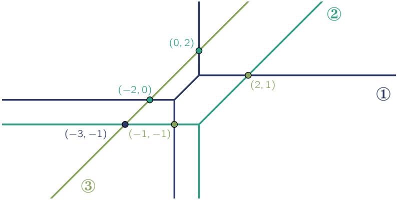
- The matrix $\widetilde{\mathcal{M}}_{\mathcal{E}\mathcal{E}} = (\widetilde{m}_{pp'})_{(p,p')\in\mathcal{E}\times\mathcal{E}}$ obtained by setting $\widetilde{m}_{pp'} = m_{pp'} - h(p' - \delta)$ is **tropically diagonally dominant**.
- Therefore its tropical right null space is reduced to $\{\mathbb{0}\}$, and thus this is also the case for $\mathcal{M}_{\mathcal{E}\mathcal{E}}$.
- Hence there does not exist $y \in \mathbb{R}^{\mathcal{E}}$ such that $\mathcal{M}_{\mathcal{E}} \odot y \nabla \mathbb{0}$.
- Finally, since $\mathcal{M}_{\mathcal{E}'}$ can be written by block as

$$\mathcal{M}_{\mathcal{E}'} = \begin{array}{c} \begin{matrix} \mathcal{E} & \quad \mathcal{E}' \setminus \mathcal{E} \end{matrix} \\ \begin{pmatrix} \mathcal{M}_{\mathcal{E}} & \quad \mathbb{0} \\ * & \quad * \end{pmatrix} \end{array},$$

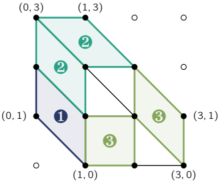
we deduce that there does also not exist $y \in \mathbb{R}^{\mathcal{E}'}$ such that $\mathcal{M}_{\mathcal{E}'} \odot y \nabla \mathbb{0}$.

Some illustrations

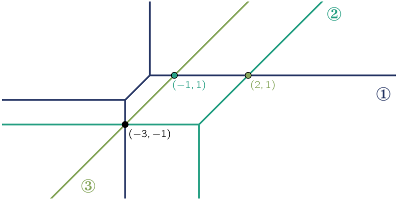
(a) The arrangement of tropical varieties of the polynomials from the system (E_1).



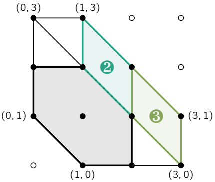
(b) The subdivision of Q associated to (E_1).



(c) The arrangement of tropical varieties of the polynomials from the system (E_2).



(d) The subdivision of Q associated to (E_2).

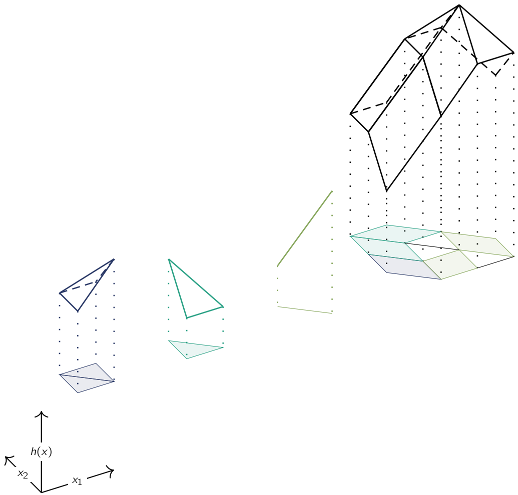


Example: The matrix associated with system (E_1) is

$$\mathcal{M}_{\mathcal{E}}^{(1)} = \begin{array}{c} \\ f_1 \\ f_2 \\ x_1 f_2 \\ x_2 f_2 \\ f_3 \\ x_1 f_3 \\ x_2 f_3 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & x_1 & x_2 & x_1^2 & x_1 x_2 & x_2^2 \end{array} \\ \begin{pmatrix} 1 & 2 & 1 & & 1 & \\ 0 & 0 & 1 & & & \\ & 0 & & 0 & 1 & \\ & & 0 & & 0 & 1 \\ & 2 & 0 & & & \\ & & & 2 & 0 & \\ & & & & 2 & 0 \end{pmatrix} \end{array} .$$

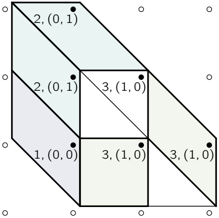
There is no finite vector in its tropical right null space and thus there is no finite solution to the equation $f(x) \nabla \mathbb{0}$.

Figure: The subdivision of Q associated to (E_1) arises from the projection of the Minkowski sum of the hypographs of the lifted Newton polytopes.



Some illustrations

Figure: The polytope $Q+\delta$, with the integer points inside the maximal dimensional cells of the decomposition of $Q+\delta$ labelled by the row content the cell they belong to.



This configuration yields the following nonsingular square submatrix of $\mathcal{M}_{\mathcal{E}}^{(1)}$

$$\mathcal{M}_{\mathcal{E}\mathcal{E}}^{(1)} = \begin{array}{r} \\ (0,0)\to f_1 \\ (1,0)\to f_3 \\ (0,1)\to f_2 \\ (2,0)\to x_1 f_3 \\ (1,1)\to x_2 f_3 \\ (0,2)\to x_2 f_2 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 \end{array} \\ \begin{pmatrix} 1 & 2 & 1 & & 1 & \\ & 2 & 0 & & & \\ 0 & 0 & 1 & & & \\ & & & 2 & 0 & \\ & & & & 2 & 0 \\ & & 0 & & 0 & 1 \end{pmatrix} \end{array}.$$

Example: The matrix associated with system (E_2) is

$$
\mathcal{M}_{\mathcal{E}}^{(2)} = \begin{array}{r} \\ f_1 \\ f_2 \\ x_1 f_2 \\ x_2 f_2 \\ f_3 \\ x_1 f_3 \\ x_2 f_3 \end{array}
\begin{array}{c} \begin{array}{cccccc} 1 & x_1 & x_2 & x_1^2 & x_1 x_2 & x_2^2 \end{array} \\
\begin{pmatrix}
1 & 4 & 1 & & 3 & \\
0 & 0 & 1 & & & \\
 & 0 & & 0 & 1 & \\
 & & 0 & & 0 & 1 \\
 & 2 & 0 & & & \\
 & & & 2 & 0 & \\
 & & & & 2 & 0
\end{pmatrix} \end{array} .
$$

The vector $y = \mathrm{ver}(-3, -1) = (0, -3, -1, -6, -4, -2)$ is finite and is in the tropical right null space of the previous matrix, hence there is a finite solution to the equation $f(x) \nabla \mathbb{0}$, which is indeed given by $(-3, -1)$.

The tropical Positivstellensatz

- Let $f^{\pm} = (f_1^{\pm}, \ldots, f_k^{\pm})$ be two collections of tropical polynomials. For $1 \leq i \leq k$, denote by $\mathcal{A}_i^{\pm}$ the support of $f_i^{\pm}$.
- Set $\rhd = (\rhd_1, \ldots, \rhd_k)$ a collection of relations, with $\rhd_i \in \{\geq, =, >\}$ for $1 \leq i \leq k$.

We denote by $f^+(x) \rhd f^-(x)$ the system

$$\max_{\alpha \in \mathcal{A}_i^+} \left(f_{i,\alpha}^+ + \langle \alpha, x \rangle\right) \rhd_i \max_{\alpha \in \mathcal{A}_{i^-}} \left(f_{i,\alpha}^- + \langle \alpha, x \rangle\right) \text{ for all } 1 \leq i \leq k$$

of unknown $x \in (\mathbb{R} \cup \{-\infty\})^n$.

The tropical Positivstellensatz

- Let $\mathcal{M}^{\pm}$ be the Macaulay matrices associated to $f^{\pm}$ – *i.e.* with entries $f^{\pm}_{i,\beta-\alpha}$. For any subset $\mathcal{E}$ of $\mathbb{Z}^n$, denote by $\mathcal{M}^{\pm}_{\mathcal{E}}$ the submatrices of $\mathcal{M}^{\pm}$ by taking only the row indices $(i,\alpha) \in [k] \times \mathbb{Z}^n$ such that the supports of the rows (i,α) of both $\mathcal{M}^{+}$ and $\mathcal{M}^{-}$ is included in $\mathcal{E}$ and the column indices given by $\mathcal{E}$.
- Finally, denote by $\mathcal{M}^{+}_{\mathcal{E}} \odot y \rhd \mathcal{M}^{-}_{\mathcal{E}} \odot y$ the following system of tropical linear inequalities:

$$\max_{\beta\in\mathcal{E}} \left(\mathcal{M}^{+}_{(i,\alpha),\beta} + y_\beta\right) \rhd_i \max_{\beta\in\mathcal{E}} \left(\mathcal{M}^{-}_{(i,\alpha),\beta} + y_\beta\right) \text{ for all } 1 \le i \le k.$$

Let $\widetilde{Q} = r_1 Q_1 + \cdots + r_k Q_k$, where $Q_i^{\pm} = \mathrm{conv}(\mathcal{A}_i^{\pm})$ and $Q_i = \mathrm{conv}(\mathcal{A}_i^{+} \cup \mathcal{A}_i^{-})$ for $i = 1, \ldots, k$, and

$$r_i = \begin{cases} \dim(Q_i^{-}) + 1 & \text{if } \rhd_i \in \{\geq, >\} \\ \max(\dim(Q_i^{-}), \dim(Q_i^{+})) + 1 & \text{if } \rhd_i \in \{=\} \end{cases} .$$

We now call **Canny-Emiris subsets** of $\mathbb{Z}^n$ associated to the pair of collections (f^{+}, f^{-}) any set $\mathcal{E}$ of the form

$$\mathcal{E} := \left(\widetilde{Q} + \delta\right) \cap \mathbb{Z}^n \ ,$$

where δ is a generic vector in $V + \mathbb{Z}^n$, with V the direction of the affine hull of $\widetilde{Q}$.

Tropical Positivstellensatz

There exists a solution $x \in \mathbb{R}^n$ to the system $f^+(x) \rhd f^-(x)$ if and only if there exists a vector $y \in \mathbb{R}^{\mathcal{E}'}$ satisfying $\mathcal{M}^+_{\mathcal{E}'} \odot y \rhd \mathcal{M}^-_{\mathcal{E}'} \odot y$, where $\mathcal{E}'$ is any subset of $\mathbb{Z}^n$ containing a nonempty Canny-Emiris subset $\mathcal{E}$ of $\mathbb{Z}^n$ associated to the pair (f^+, f^-).

Corollary: The inclusion of basic tropical semialgebraic sets can be reduced to solving a set of tropical linear (in)equalities.

The inclusion problem for tropical semialgebraic sets

The tropical Positivstellensatz provides an effective way to check the inclusion of two tropical basic semialgebraic sets, as checking whether the following implication holds

$$\left\{\begin{array}{ccc} f_1^+(x) & \geq & f_1^-(x) \\ & \vdots & \\ f_k^+(x) & \geq & f_k^-(x) \end{array}\right. \implies f_{k+1}^+(x) \geq f_{k+1}^-(x) \quad \forall x \in \mathbb{R}^n \ .$$

is equivalent to showing the unfeasibility of the following system

$$\left\{\begin{array}{ccc} f_1^+(x) & \geq & f_1^-(x) \\ & \vdots & \\ f_k^+(x) & \geq & f_k^-(x) \\ f_{k+1}^+(x) & < & f_{k+1}^-(x) \end{array}\right. .$$

The certificate of unfeasibility will be given by a policy of the minimizer player in a mean payoff game.

The Shapley-Folkman Lemma

Let $A_1, \ldots, A_k \subseteq \mathbb{R}^n$, and let

$$x \in \sum_{i=1}^{k} \mathrm{conv}(A_i) \quad .$$

Then there is an index set $I \subseteq \{1, \ldots, k\}$ with $|I| \leq n$ such that

$$x \in \sum_{i \in I} \mathrm{conv}(A_i) + \sum_{i \in \{1,\ldots,k\} \setminus I} A_i \quad .$$

Corollary: If $\sum_{i=1}^{k} \mathrm{conv}(A_i)$ has (affine) dimension $d < n$, then the index set I can be choosen such that $|I| \leq d$.

III - Mean payoff games and tropical linear systems

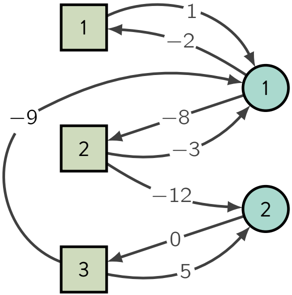
Mean payoff games

Mean payoff games (See **Gillette (1957)**,
Gurvich, Karzanov, Khachiyan (1988),
Zwick, Patterson (1996)):

- $G = (I \sqcup J, E)$ a (finite) oriented weighted bipartite graph;
- game with two players Min and Max: each turn, from the current state $j_0 \in J$, player Min chooses a state $i_0 \in I$ such that (j_0, i_0) is an arc of G with weight $-a_{i_0 j_0}$ and obtains a payment of $a_{i_0 j_0}$ from player Max, then player Max from the state $i_0 \in I$, chooses the next state $j_1 \in J$ along an arc (i_0, j_1) with weight $b_{i_0 j_1}$, and receives in turn a payment of $b_{i_0 j_1}$ from player Min;
- the winner is the player who gets the highest average payment per turn;
- set $A = (a_{ij})_{(i,j)\in I\times J}$ et $B = (b_{ij})_{(i,j)\in I\times J}$.

Mean payoff games

Example : Let G be the following graph:



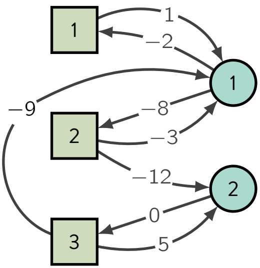
One has $A = \begin{pmatrix} 2 & -\infty \\ 8 & -\infty \\ -\infty & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -\infty \\ -3 & -12 \\ -9 & 5 \end{pmatrix}$.

Theorem [Akian, Gaubert, Guterman (2012)] : For all $j \in J$, player Max has a winning positional strategy for the *mean pay-off game* given by the payment matrices A and B starting at the initial position j iff there exists a solution $y \in (\mathbb{R} \cup \{-\infty\})^J$ of the tropical matrix inequality $A \odot y \leq B \odot y$ such that $y_j \neq \mathbb{0}$.

The winning initial positions correspond to the support of the solutions of the inequality $A \odot y \leq B \odot y$.

Mean payoff games and tropical linear systems

In the previous example,



one has $A \odot y \leq B \odot y \iff \begin{cases} 2 + y_1 & \leq & 1 + y_1 \\ 8 + y_1 & \leq & \max(-3 + y_1, -12 + y_2) \\ y_2 & \leq & \max(-9 + y_1, 5 + y_2). \end{cases}$

The first inequality entails that every solution $y \in (\mathbb{R} \cup \{-\infty\})^2$ must satisfy $y_1 = \mathbb{0}$, and then the other two inequalities are satisfied for all values of $y_2 \in \mathbb{R} \cup \{-\infty\}$. This thus means that (1) is a losing initial position for player Max, while (2) is a winning initial position.

The Shapley operator

- **Shapley operator** associated to a mean payoff game

$$T : \begin{array}{ccc} (\mathbb{R} \cup \{\pm\infty\})^J & \longrightarrow & (\mathbb{R} \cup \{\pm\infty\})^J \\ y = (y_j)_{j \in J} & \longmapsto & \left(\min_{i \in I} -a_{ij} + \left(\max_{k \in J} b_{ik} + y_k \right) \right)_{j \in J} \end{array}$$

- **value** of the game: $\chi(T) = \lim_{N \to +\infty} \dfrac{T^N(0)}{N}$

Feasibility criterion [Akian, Gaubert, Guterman (2012)]:

$$\exists y \in \mathbb{R}^J \text{ such that } A \odot y \leq B \odot y \quad \Longleftrightarrow \quad \min_{j \in J} \chi_j(T) \geq 0 \ .$$

Link with nonlinear eigenvalue theory:

$$\begin{aligned}
&\min\{\chi_j(T) : j \in J\} \\
&= \sup\{\lambda \in \mathbb{R} : \exists u \in \mathbb{R}^J, T(u) \geq \lambda + u\} \\
&= \inf\{\lambda \in \mathbb{R} \cup \{+\infty\} : \exists u \in (\mathbb{R} \cup \{+\infty\})^J, u \not\equiv +\infty, T(u) \leq \lambda + u\} \\
&= \inf\{\lambda \in \mathbb{R} \cup \{+\infty\} : \exists u \in (\mathbb{R} \cup \{+\infty\})^J, u \not\equiv +\infty, T(u) = \lambda + u\} \ .
\end{aligned}$$

In particular $\chi(T) \equiv \lambda \in \mathbb{R}$ iff the nonlinear eigenproblem

$$T(u) = \lambda + u$$

has a solution $(\lambda, u) \in \mathbb{R} \times \mathbb{R}^J$.

IV - Solving tropical polynomial systems with mean payoff games

The existence of a polynomial time algorithm to solve mean payoff games is an open problem (see Gurvich, Karzanov, Khachiyan (1988)), but there exist practically fast methods (value/policy iteration algorithms).

For a Shapley operator $T : (\mathbb{R} \cup \{+\infty\})^J \to (\mathbb{R} \cup \{+\infty\})^J$, define the Krasnoselskii-Mann damped Shapley operator T_{KM} by $T_{\mathsf{KM}}(u) = \frac{u+T(u)}{2}$ for all $u \in (\mathbb{R} \cup \{+\infty\})^J$. Then $\chi(T_{\mathsf{KM}}) = \frac{\chi(T)}{2}$

The value iteration algorithm

Value iteration algorithm with widening:

- Iterate the operator $\tilde{T} : u \mapsto u \wedge T_{\mathsf{KM}}(u)$ to construct the sequence $(u_N)_{N\in\mathbb{N}} = (\tilde{T}^N(0))_{N\in\mathbb{N}}$.
- Carry on the iteration until $u_{N+1} = u_N$ or $u_{N+1} \ll u_N$.
- If $u_{N+1} = u_N$ then $\min_{j\in J} \chi_j(T) \geq 0$ (feasible case).
- If $u_{N+1} \ll u_N$ then $\min_{j\in J} \chi_j(T) < 0$ (infeasible case).
- Additional unfeasibility check with a widening step which maps the stationary coordinates of u_N to $+\infty$.
- Prescribed timeout N^*.

Remark: The timeout can be chosen of the same order as the standard Zwick-Paterson algorithm, though the present algorithm is faster on average. Besides, the Krasnoselskii-Mann damping is crucial for the termination of the algorithm.

Correction and termination of the value iteration algorithm

The present algorithm applied to the Krasnoselskii-Mann damped Shapley operator T_{KM} correctly decides the feasibility of a tropical linear system with integer coefficients in $N^* = \mathcal{O}(|J|^2 W)$ iterations where W is an upper bound on the maximal non $-\infty$ coefficients of A and B.

The value iteration algorithm

- The present algorithm remains correct and terminates when executed with an η-approximation of T for sufficiently small approximation errors η.
- The cost of each evaluation of the operator T is linear in the number of nonzero entries of the Macaulay matrix.
- The total number of iterations performed by the algorithm is pseudo-polynomial.

Short solutions of tropical polynomial systems

Let $f^{\pm} = (f_1^{\pm}, \dots, f_k^{\pm})$ be two collections of tropical polynomials and let $d = \max_{1 \le i \le k} \deg(f_i^{\pm})$ and $W = \max_{1 \le i \le k} \|f_i^{\pm}\|_\infty$. Then:

Short solutions property

If the system $f^+(x) \geq f^-(x)$ has a solution in $\mathbb{R}^n$, then:

- there exists a solution in a $\|\cdot\|_\infty$-ball of radius $2n(2d)^{n-1}W$ centered at point 0;
- moreover, if all the coefficients of the polynomials $f_i^{\pm}$ are integers, then this solution can be chosen such that its coordinates are rational numbers with a denominator bounded above by $(2d)^n$.

The dichotomic search method

- Solve the system
$$\begin{cases} f^+(x) \rhd f^-(x) \\ a \le x_1 \le b \end{cases}$$
for varying values of a and b.
- If $|b - a| < \frac{1}{(2d)^n}$ then one can deduce the first coordinate of a solution.
- Fix the value of x_1 and repeat with $x_2, \dots, x_n$.

The dichotomic search method returns a rational solution of this system (or decides that there is none) in $\mathcal{O}\big(\log(n(2d)^{2n-1}W)\big)$ calls to a weak mean payoff oracle.

The path-following method

- Solve the system

$$f^{+}(\zeta, x_2, \dots, x_n) \rhd f^{-}(\zeta, x_2, \dots, x_n)$$

  for varying values of ζ.
- Linearize the above system and consider the associated mean payoff game with its shapley operator T_ζ.
- The spectral function $\phi : \zeta \mapsto \min_{j \in J} \chi_j(T_\zeta)$ is a Lipschitz continuous piecewise affine function.
- The superlevel set $\{\zeta \in \mathbb{R} : \phi(\zeta) \geq 0\}$ yields the projection of the solution set onto the first coordinate.
- Computing ϕ will be done with a pivoting algorithm.

The path-following method

- More generally, for $T_\zeta = A^\sharp_\zeta B_\zeta$ with A_ζ and B_ζ piecewise-affine in ζ, one try to find a solution of the nonlinear eigenproblem

$$T_\zeta(u(\zeta)) = \lambda(\zeta) + u(\zeta)$$

  that is piecewise affine in ζ.
- The map $\zeta \mapsto \lambda(\zeta)$ is continuous and coincides with the spectral function. **However,** $\zeta \mapsto u(\zeta)$ **might have some discontinuity points.**
- The discontinuity points arise at values of ζ such that the nonlinear eigenvector $u(\zeta)$ is **not unique**.
- The uniqueness of the solution to the eigenproblem relies on properties of the saturation graph of the operator T_ζ.

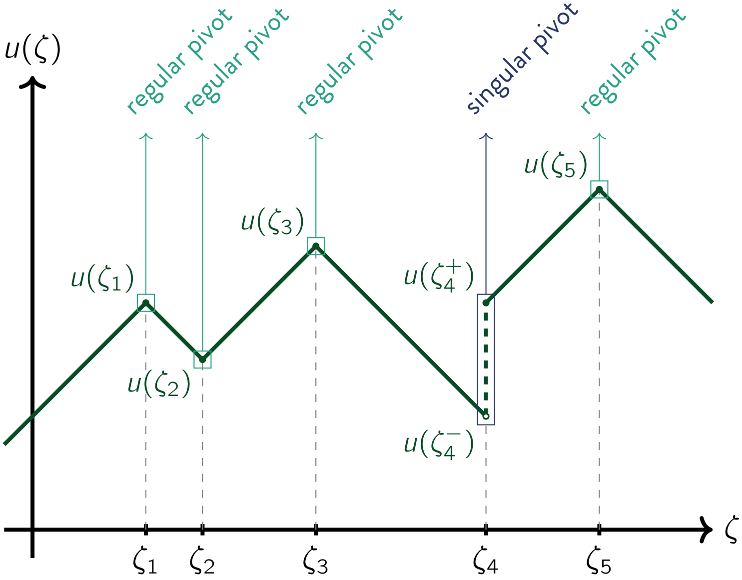
The path-following method

- For $\delta \in \mathbb{R}^{I \times J}$, one can consider the perturbed operator $T_{\delta,\zeta}$ defined by

$$T_{\delta,\zeta}(u) = \left(\min_{i \in I} -a_{ij}(\zeta) + \left(\max_{k \in J} b_{ik}(\zeta) + \delta_{ik} + u_k \right) \right)_{j \in J} .$$

- Two polyhedral complexes in the space of parameters $(\delta, \zeta) \in \mathbb{R}^{I \times J} \times \mathbb{R}$ residing at the core of the termination of the path-following algorithm: a uniqueness complex for the nonlinear eigenvector $u(\delta, \zeta)$, which can then be refined into a linearity complex.
- For a generic choice of δ, this proves the finiteness of the number of pivoting points.

The path-following method



The path-following method

- Moreover, the nonlinear eigenspace at a singular pivot ζ is generically a polyhedral line.
- The end points correspond to the eigenvectors that admit an affine left or right continuation.
- One can then explore the eigenspace $\mathrm{Eig}(T_\zeta)$ with an auxiliary pivoting algorithm relying on directional derivatives of the Shapley operator T_ζ.

The path-following method

$\mathrm{Eig}(T_{\zeta_1}) =$ $u(\zeta_1)$

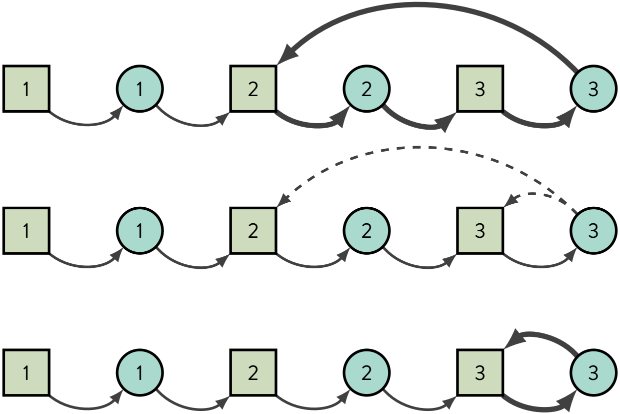
$\mathrm{Eig}(T_{\zeta_2}) =$ $u(\zeta_2)$

$\mathrm{Eig}(T_{\zeta_3}) =$ $u(\zeta_3)$

$\mathrm{Eig}(T_{\zeta_4}) =$

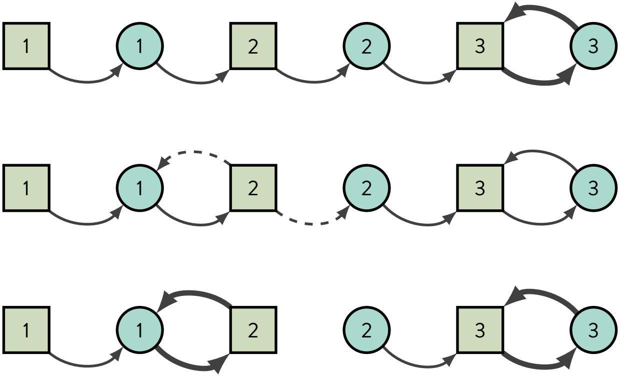
$\mathrm{Eig}(T_{\zeta_5}) =$ $u(\zeta_5)$

The path-following method



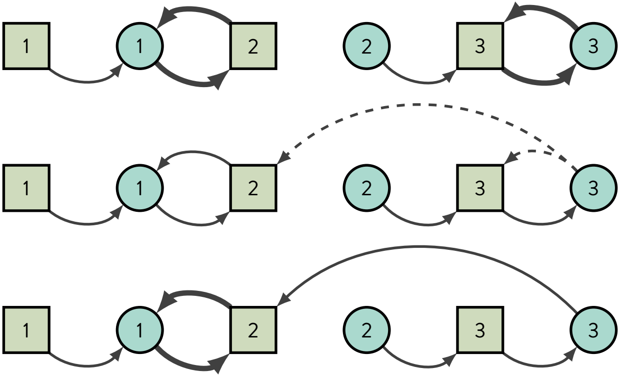
A regular pivoting of the saturation graph

The path-following method



A singular pivoting leading to the appearance of a second critical cycle

The path-following method



The disappearing of the second critical cycle

Publications

Python implementation of the algorithm available at:
`https://gitlab.inria.fr/abereau/tropical-polynomial-system-solving`

Publications and preprints:

- Akian, B., Gaubert. The Tropical Nullstellensatz and Positivstellensatz for Sparse Polynomial Systems. In *Proceedings of ISSAC '23* (2023)
- Akian, B., Gaubert. The Nullstellensatz and Positivstellensatz for sparse tropical polynomial systems. *arXiv:2312.05859* (2023)
- Akian, B., Gaubert. Eigenvalue methods for sparse tropical polynomial systems. In *Mathematical Software – ICMS 2024* (2024)
- **In progress:** Akian, B., Gaubert. The Krasnoselskii-Mann iteration for fixed-point free polyhedral nonexpansive mappings.

Perspectives

- Are there Canny-Emiris/determinantal type formulae for the tropical resultant?
- Can the degree bound be improved in the Positivstellensatz (no tight example found yet)?
- Explicit complexity bounds for the path-following method (termination proven but without explicit bounds)?
- Can singular pivot generically be avoided (similar to the discriminant variety for polynomial homotopy methods)?

Thank you for your attention!