The Nullstellensatz and Positivstellensatz for Sparse Tropical Polynomial Systems

Marianne Akian, Antoine Béreau, Stéphane Gaubert

CMAP, CNRS, École polytechnique, Institut Polytechnique de Paris, Inria

July 27th, 2023

ISSAC 2023, Tromsø

- Given system of tropical polynomial equations or inequations, how to check the existence of a solution in $\mathbb{R}^n$.

- Given system of tropical polynomial equations or inequations, how to check the existence of a solution in $\mathbb{R}^n$.
- Main tools in the classical setting include the theory of resultants, Macaulay matrices and effective Null- and Positivstellensatz.

- Given system of tropical polynomial equations or inequations, how to check the existence of a solution in $\mathbb{R}^n$.
- Main tools in the classical setting include the theory of resultants, Macaulay matrices and effective Null- and Positivstellensatz.
- In this talk, we develop the tropical analog of the sparse Null- and Positivstellensatz, and their link with mean payoff games.

I - Tropical algebra and tropical polynomials

- **Tropical semiring** $\mathbb{R}_\infty = (\mathbb{R} \cup \{-\infty\}, \oplus, \odot)$ with
  - addition $\oplus := \max$;
  - multiplication $\odot := +$;
  - zero element $\mathbb{0} := -\infty$;
  - unit element $\mathbb{1} := 0$.

- **Tropical semiring** $\mathbb{R}_\infty = (\mathbb{R} \cup \{-\infty\}, \oplus, \odot)$ with
  - addition $\oplus := \max$;
  - multiplication $\odot := +$;
  - zero element $\mathbb{0} := -\infty$;
  - unit element $\mathbb{1} := 0$.
- Satisfies the usual properties of a field except **no additive inverse**.

- **Tropical semiring** $\mathbb{R}_\infty = (\mathbb{R} \cup \{-\infty\}, \oplus, \odot)$ with
  - addition $\oplus := \max$;
  - multiplication $\odot := +$;
  - zero element $\mathbb{0} := -\infty$;
  - unit element $\mathbb{1} := 0$.
- Satisfies the usual properties of a field except **no additive inverse**.
- Tropical operations can be extended to vectors and matrices with coefficients in $\mathbb{R}_\infty$ allowing us to perform **tropical linear algebra**.

- A **formal tropical polynomial** p in n variables is a map

$$\begin{array}{rcl} \mathbb{Z}^n & \longrightarrow & \mathbb{R}_\infty \\ \alpha & \longmapsto & p_\alpha \end{array}$$

  such that $p_\alpha \neq \mathbb{0}$ for finitely many $\alpha \in \mathbb{Z}^n$. We denote $p = \bigoplus_{\alpha \in \mathbb{Z}^n} p_\alpha X^\alpha$.

- A **formal tropical polynomial** p in n variables is a map

$$\begin{array}{ccc} \mathbb{Z}^n & \longrightarrow & \mathbb{R}_\infty \\ \alpha & \longmapsto & p_\alpha \end{array}$$

  such that $p_\alpha \neq \mathbb{0}$ for finitely many $\alpha \in \mathbb{Z}^n$. We denote $p = \bigoplus_{\alpha \in \mathbb{Z}^n} p_\alpha X^\alpha$.
- **Support** of p: $\mathrm{supp}(p) := \{\alpha \in \mathbb{Z}^n : p_\alpha \neq \mathbb{0}\}$

- A **formal tropical polynomial** p in n variables is a map

$$\begin{array}{ccc} \mathbb{Z}^n & \longrightarrow & \mathbb{R}_\infty \\ \alpha & \longmapsto & p_\alpha \end{array}$$

  such that $p_\alpha \neq \mathbb{0}$ for finitely many $\alpha \in \mathbb{Z}^n$. We denote $p = \bigoplus_{\alpha \in \mathbb{Z}^n} p_\alpha X^\alpha$.
- **Support** of p: $\mathrm{supp}(p) := \{\alpha \in \mathbb{Z}^n : p_\alpha \neq \mathbb{0}\}$
- **Polynomial function** associated to p:

$$\hat{p} : \left\{ \begin{array}{ccc} \mathbb{R}^n & \longrightarrow & \mathbb{R}_\infty \\ x & \longmapsto & \hat{p}(x) := \max_{\alpha \in \mathcal{A}}(p_\alpha + \langle x, \alpha \rangle) \end{array} \right.$$

  with $\mathcal{A} = \mathrm{supp}(p)$

- A **formal tropical polynomial** p in n variables is a map

$$\begin{array}{ccc} \mathbb{Z}^n & \longrightarrow & \mathbb{R}_\infty \\ \alpha & \longmapsto & p_\alpha \end{array}$$

  such that $p_\alpha \neq \mathbb{0}$ for finitely many $\alpha \in \mathbb{Z}^n$. We denote $p = \bigoplus_{\alpha \in \mathbb{Z}^n} p_\alpha X^\alpha$.
- **Support** of p: $\mathrm{supp}(p) := \{\alpha \in \mathbb{Z}^n : p_\alpha \neq \mathbb{0}\}$
- **Polynomial function** associated to p:

$$\hat{p} : \left\{ \begin{array}{ccc} \mathbb{R}^n & \longrightarrow & \mathbb{R}_\infty \\ x & \longmapsto & \hat{p}(x) := \max_{\alpha \in \mathcal{A}}(p_\alpha + \langle x, \alpha \rangle) \end{array} \right.$$

  with $\mathcal{A} = \mathrm{supp}(p)$

Remark : A tropical polynomial function is a convex, piecewise affine function with integer slopes.

A point $x \in \mathbb{R}^n_\infty$ is a **root** of a polynomial p whenever the maximum in the expression

$$\hat{p}(x) = \bigoplus_{\alpha \in \mathcal{A}} p_\alpha \odot x^{\odot \alpha} = \max_{\alpha \in \mathcal{A}} (p_\alpha + \langle x, \alpha \rangle)$$

is attained for at least two distinct values of α. This is denoted as $p(x) \nabla \mathbb{0}$.

A point $x \in \mathbb{R}^n_\infty$ is a **root** of a polynomial p whenever the maximum in the expression

$$\hat{p}(x) = \bigoplus_{\alpha \in \mathcal{A}} p_\alpha \odot x^{\odot \alpha} = \max_{\alpha \in \mathcal{A}} (p_\alpha + \langle x, \alpha \rangle)$$

is attained for at least two distinct values of α. This is denoted as $p(x) \nabla \mathbb{0}$.

Exemple : Consider the tropical polynomial $f_1 = 1 \oplus 2x_1 \oplus 1x_2 \oplus 1x_1x_2$. Then:

A point $x \in \mathbb{R}^n_\infty$ is a **root** of a polynomial p whenever the maximum in the expression

$$\hat{p}(x) = \bigoplus_{\alpha \in \mathcal{A}} p_\alpha \odot x^{\odot \alpha} = \max_{\alpha \in \mathcal{A}} (p_\alpha + \langle x, \alpha \rangle)$$

is attained for at least two distinct values of α. This is denoted as $p(x) \nabla \mathbb{0}$.

Exemple : Consider the tropical polynomial $f_1 = 1 \oplus 2x_1 \oplus 1x_2 \oplus 1x_1x_2$. Then:

- $(0, 2)$ is a root of f_1 since the maximum of $\hat{f}_1(0, 2) = 3$ is attained simultaneously by the monomials $1x_2$ and $1x_1x_2$;

A point $x \in \mathbb{R}^n_\infty$ is a **root** of a polynomial p whenever the maximum in the expression

$$\hat{p}(x) = \bigoplus_{\alpha \in \mathcal{A}} p_\alpha \odot x^{\odot \alpha} = \max_{\alpha \in \mathcal{A}} (p_\alpha + \langle x, \alpha \rangle)$$

is attained for at least two distinct values of α. This is denoted as $p(x) \nabla \mathbb{0}$.

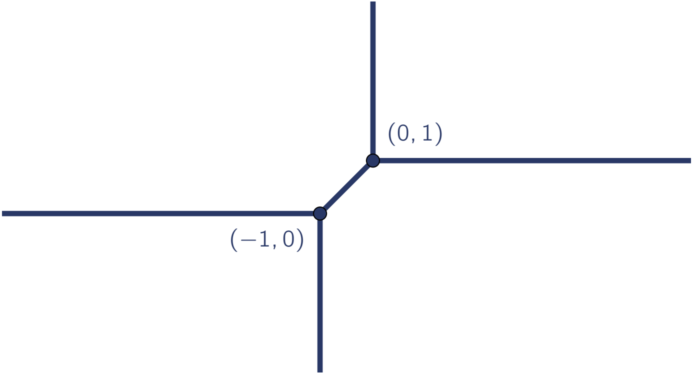
Exemple : Consider the tropical polynomial $f_1 = 1 \oplus 2x_1 \oplus 1x_2 \oplus 1x_1x_2$. Then:

- $(0, 2)$ is a root of f_1 since the maximum of $\hat{f}_1(0, 2) = 3$ is attained simultaneously by the monomials $1x_2$ and $1x_1x_2$;
- $(-1, 1)$ is not a root of f_1 since the maximum $\hat{f}_1(-1, 1) = 2$ is attained only by the monomial $1x_2$.

The **tropical hypersurface** associated to a tropical polynomial p is the set of its roots. It coincides with the non-differentiability locus of the fonction $\hat{p}$.

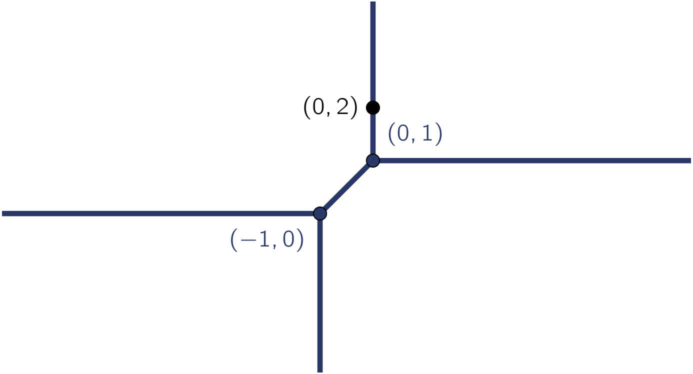
The **tropical hypersurface** associated to a tropical polynomial p is the set of its roots. It coincides with the non-differentiability locus of the fonction $\hat{p}$.

Example : The tropical hypersurface associated to the polynomial $f_1 = 1 \oplus 2x_1 \oplus 1x_2 \oplus 1x_1x_2$ is the following:



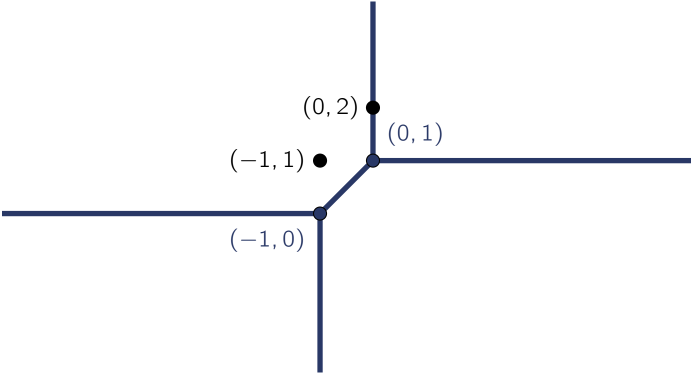
The **tropical hypersurface** associated to a tropical polynomial p is the set of its roots. It coincides with the non-differentiability locus of the fonction $\hat{p}$.

Example : The tropical hypersurface associated to the polynomial $f_1 = 1 \oplus 2x_1 \oplus 1x_2 \oplus 1x_1x_2$ is the following:



The **tropical hypersurface** associated to a tropical polynomial p is the set of its roots. It coincides with the non-differentiability locus of the fonction $\hat{p}$.

Example : The tropical hypersurface associated to the polynomial $f_1 = 1 \oplus 2x_1 \oplus 1x_2 \oplus 1x_1x_2$ is the following:



Likewise, $y \in \mathbb{R}_\infty^m$ is said to be in the **tropical right null space** or **kernel** of a $\ell \times m$ matrix $A = (a_{ij})$ whenever for all $1 \le i \le \ell$, the maximum in the expression

$$\bigoplus_{j=1}^{m} a_{ij} \odot y_j = \max_{1 \le j \le m} (a_{ij} + y_j)$$

is achieved at least twice. This is also denoted as $A \odot y \nabla \mathbb{0}$.

Likewise, $y \in \mathbb{R}_\infty^m$ is said to be in the **tropical right null space** or **kernel** of a $\ell \times m$ matrix $A = (a_{ij})$ whenever for all $1 \leq i \leq \ell$, the maximum in the expression

$$\bigoplus_{j=1}^{m} a_{ij} \odot y_j = \max_{1 \leq j \leq m} (a_{ij} + y_j)$$

is achieved at least twice. This is also denoted as $A \odot y \nabla \mathbb{0}$.

More on tropical geometry: D. Maclagan and B. Sturmfels. *Introduction to Tropical Geometry*. Graduate Studies in Mathematics. American Mathematical Society, 2015.

II - Position of the problem

In the following, we fix a collection $f = (f_1, \ldots, f_k)$ of k formal tropical polynomials in n variables, with respective supports $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_k)$ and degrees $(d_1, \ldots, d_k)$.

In the following, we fix a collection $f = (f_1, \ldots, f_k)$ of k formal tropical polynomials in n variables, with respective supports $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_k)$ and degrees $(d_1, \ldots, d_k)$.

Problem: Decide whether there is a common tropical zero $x \in \mathbb{R}^n$, that is such that $f_i(x) \,\nabla\, \mathbb{0}$ for all $1 \leq i \leq n$.

In the following, we fix a collection $f = (f_1, \ldots, f_k)$ of k formal tropical polynomials in n variables, with respective supports $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_k)$ and degrees $(d_1, \ldots, d_k)$.

Problem: Decide whether there is a common tropical zero $x \in \mathbb{R}^n$, that is such that $f_i(x) \nabla \mathbb{0}$ for all $1 \leq i \leq n$.

In other words, we ask the question of the nonemptyness of the intersection of $\mathbb{R}^n$ with the tropical prevariety given by the intersection of the tropical hypersurfaces associated to the f_i.

In the following, we fix a collection $f = (f_1, \ldots, f_k)$ of k formal tropical polynomials in n variables, with respective supports $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_k)$ and degrees $(d_1, \ldots, d_k)$.

Problem: Decide whether there is a common tropical zero $x \in \mathbb{R}^n$, that is such that $f_i(x) \nabla \mathbb{0}$ for all $1 \leq i \leq n$.

In other words, we ask the question of the nonemptyness of the intersection of $\mathbb{R}^n$ with the tropical prevariety given by the intersection of the tropical hypersurfaces associated to the f_i.

Remark: The same question exists for solution in $\mathbb{R}^n_\infty$. It reduces to the $\mathbb{R}^n$ case by considering the support of the solutions.

Figure: The arrangement of tropical varieties of the polynomials from the system

$$(E_1) : \begin{cases} f_1 = 1 \oplus 2x_1 \oplus 1x_2 \oplus 1x_1x_2 \\ f_2 = 0 \oplus 0x_1 \oplus 1x_2 \\ f_3 = 2x_1 \oplus 0x_2 \ , \end{cases} .$$

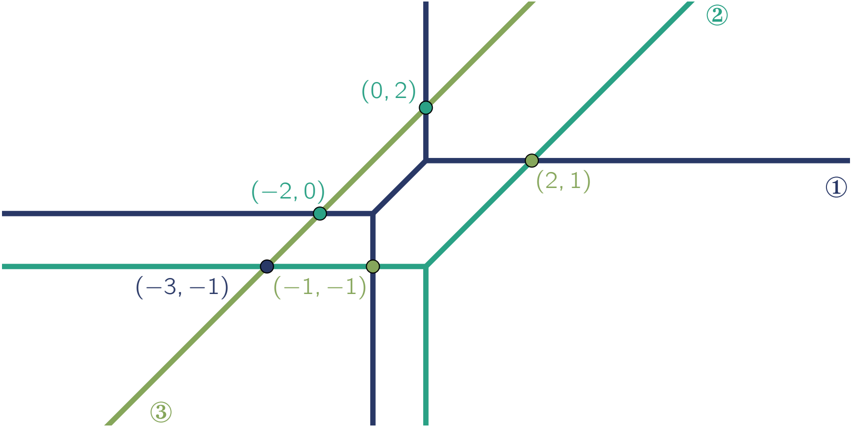
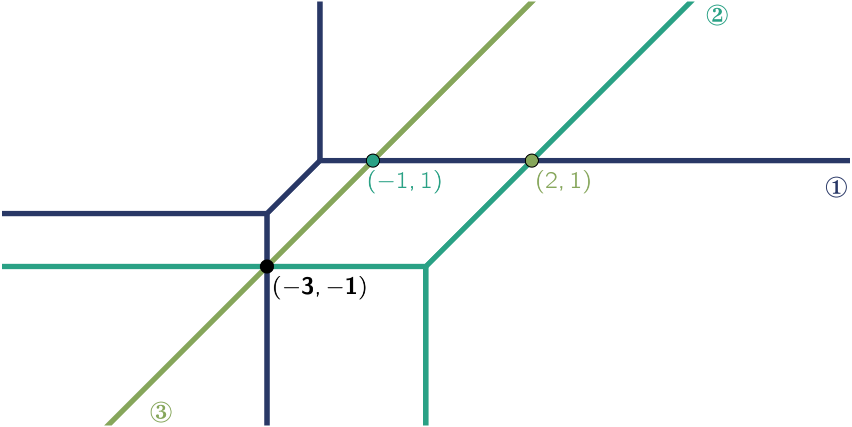


Figure: The arrangement of tropical varieties of the polynomials from the system

$$(E_2): \begin{cases} f_1 = 1 \oplus 4x_1 \oplus 1x_2 \oplus 3x_1x_2 \\ f_2 = 0 \oplus 0x_1 \oplus 1x_2 \\ f_3 = 2x_1 \oplus 0x_2 \ , \end{cases} .$$



Link with classical varieties:

- Kapranov’s theorem
- The Fundamental Theorem of Tropical Algebraic Geometry

Link with classical varieties:

- Kapranov's theorem
- The Fundamental Theorem of Tropical Algebraic Geometry

Varied applications:

- celestial mechanics (Hampton, Moeckel)
- max-out networks (Montúfar, Ren, Zhang)
- chemical reaction networks (Dickenstein, Feliu, Radulescu, Shiu)
- emergency call center (Akian, Boyer, Gaubert)

The **Macaulay matrix** associated to f is the (infinite) matrix $\mathcal{M} = (m_{(i,\alpha),\beta})$ indexed by $([n] \times \mathbb{Z}^n) \times \mathbb{Z}^n$, where $m_{(i,\alpha),\beta}$ corresponds to the coefficient of X^β in the polynomial $X^\alpha f_i$.

$$\mathcal{M} = \begin{array}{c} \\ f_1 \\ x_1 f_1 \\ \vdots \\ x^\alpha f_i \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & x_1 & \cdots & x^\beta & \cdots \end{array} \\ \begin{pmatrix} * & * & \cdots & * & \cdots \\ * & * & \cdots & * & \cdots \\ \vdots & \vdots & \ddots & \vdots & \\ * & * & \cdots & f_{i,\beta-\alpha} & \cdots \\ \vdots & \vdots & & \vdots & \ddots \end{pmatrix} \end{array}$$

- One approch to the **Nullstellensatz** is a linearization method reducing the search of a solution of a polynomial system to the search of nonzero elements in the kernel of the Macaulay matrix.

- One approch to the **Nullstellensatz** is a linearization method reducing the search of a solution of a polynomial system to the search of nonzero elements in the kernel of the Macaulay matrix.
- A finite subset $\mathcal{E}$ of $\mathbb{Z}^n$ yields a (finite) submatrix $\mathcal{M}_{\mathcal{E}}$ of $\mathcal{M}$ obtained by taking only the rows whose support is included in $\mathcal{E}$ and the columns indexed by $\mathcal{E}$.

- One approch to the **Nullstellensatz** is a linearization method reducing the search of a solution of a polynomial system to the search of nonzero elements in the kernel of the Macaulay matrix.
- A finite subset $\mathcal{E}$ of $\mathbb{Z}^n$ yields a (finite) submatrix $\mathcal{M}_{\mathcal{E}}$ of $\mathcal{M}$ obtained by taking only the rows whose support is included in $\mathcal{E}$ and the columns indexed by $\mathcal{E}$.
- For $\mathcal{E} = \{\alpha \in \mathbb{N}^n : \alpha_1 + \cdots + \alpha_n \leq N\}$, we denote $\mathcal{M}_N := \mathcal{M}_{\mathcal{E}}$.

Conjecture [Grigoriev (2012)]: There exists a finite integer N such that

$$\exists x \in \mathbb{R}^n \text{ such that } f_i(x) \,\nabla\, \mathbb{0} \text{ for } i = 1, \ldots, k$$

$$\Longleftrightarrow$$

$$\exists y \in \mathbb{R}^m \text{ such that } \mathcal{M}_N \odot y \,\nabla\, \mathbb{0} \text{ with } m = \binom{N+n}{n} \ .$$

Conjecture [Grigoriev (2012)]: There exists a finite integer N such that

$$\exists x \in \mathbb{R}^n \text{ such that } f_i(x) \nabla \mathbb{0} \text{ for } i = 1, \dots, k$$

$$\iff$$

$$\exists y \in \mathbb{R}^m \text{ such that } \mathcal{M}_N \odot y \nabla \mathbb{0} \text{ with } m = \binom{N+n}{n} \ .$$

Answer:

- **Grigoriev, Podolskii (2018):** true for

$$N = (n+2)(d_1 + \cdots + d_k) \ .$$

Conjecture [Grigoriev (2012)]: There exists a finite integer N such that

$$\exists x \in \mathbb{R}^n \text{ such that } f_i(x) \mathrel{\nabla} \mathbb{0} \text{ for } i = 1, \ldots, k$$

$$\iff$$

$$\exists y \in \mathbb{R}^m \text{ such that } \mathcal{M}_N \odot y \mathrel{\nabla} \mathbb{0} \text{ with } m = \binom{N+n}{n} \ .$$

Answer:

- **Grigoriev, Podolskii (2018):** true for

$$N = (n+2)(d_1 + \cdots + d_k) \ .$$

- **Akian, B., Gaubert (2023):** true for

$$N = d_1 + \cdots + d_k - 1$$

(and even $N = d_1 + \cdots + d_k - n$ in most cases) + adapted approch for the case of sparse polynomials.

III - The tropical Nullstellensatz for sparse polynomial systems

This work improves on Grigoriev and Podolskii's result by taking into account the sparse structure of the polynomials, and connects the tropical Nullstellensatz with classical elimination theory. In particular, it relies on a construction by Canny and Emiris (1993) and Sturmfels (1994).

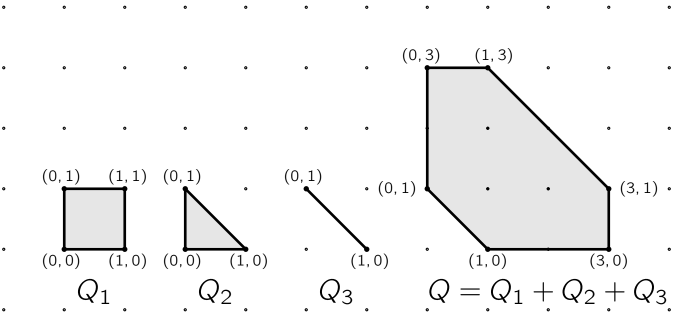
This work improves on Grigoriev and Podolskii's result by taking into account the sparse structure of the polynomials, and connects the tropical Nullstellensatz with classical elimination theory. In particular, it relies on a construction by Canny and Emiris (1993) and Sturmfels (1994).

This results in an improved truncation degree (we even recover the classical Macaulay bound whenever $k = n + 1$) and allows us to deal better with sparse polynomials.

- For $1 \leq i \leq k$, $Q_i := \mathrm{conv}(\mathcal{A}_i)$ is the **Newton polytope** of f_i.

- For $1 \leq i \leq k$, $Q_i := \text{conv}(\mathcal{A}_i)$ is the **Newton polytope** of f_i.

Example: The Newton polytopes associated to both system (E_1) and system (E_2) and their Minkowski sum are as follow.



- **Canny-Emiris set** associated to f: $\mathcal{E} = (Q + \delta) \cap \mathbb{Z}^n$ with δ a generic vector in the linear space directing the affine hull of Q.

- **Canny-Emiris set** associated to f: $\mathcal{E} = (Q + \delta) \cap \mathbb{Z}^n$ with δ a generic vector in the linear space directing the affine hull of Q.

Example: Considering again the systems (E_1) and (E_2), for

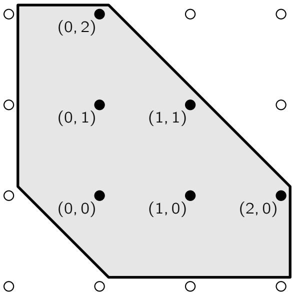
$$\delta = (-1 + \varepsilon, -1 + \varepsilon)$$

with $\varepsilon > 0$ sufficiently small, we obtain the Canny-Emiris set

$$\mathcal{E} := (Q + \delta) \cap \mathbb{Z}^n = \{(0,0), (1,0), (0,1), (2,0), (1,1), (0,2)\}$$

corresponding to the set of monomials $\{1, x_1, x_2, x_1^2, x_1x_2, x_2^2\}$.

Figure: The polytope $Q + \delta$ with $\delta = (-0.9, -0.9)$.



Tools for the proof of the result

- The upper hull of the lifted support $\{(\alpha, f_{i,\alpha}) : \alpha \in \mathcal{A}_i\}$ is the graph of a function h_i with support Q_i.

Tools for the proof of the result

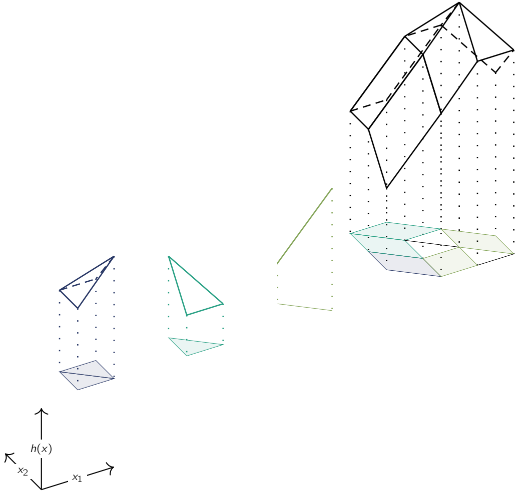
- The upper hull of the lifted support $\{(\alpha, f_{i,\alpha}) : \alpha \in \mathcal{A}_i\}$ is the graph of a function h_i with support Q_i.
- If $h := h_1 \square \cdots \square h_k$ where $\square$ denotes the sup-convolution, then $\text{hypo}(h) = \text{hypo}(h_1) + \cdots + \text{hypo}(h_k)$ and moreover the supports of h is $Q = Q_1 + \cdots + Q_k$.

Tools for the proof of the result

- The upper hull of the lifted support $\{(\alpha, f_{i,\alpha}) : \alpha \in \mathcal{A}_i\}$ is the graph of a function h_i with support Q_i.
- If $h := h_1 \,\square \cdots \square\, h_k$ where $\square$ denotes the sup-convolution, then $\mathrm{hypo}(h) = \mathrm{hypo}(h_1) + \cdots + \mathrm{hypo}(h_k)$ and moreover the supports of h is $Q = Q_1 + \cdots + Q_k$.
- The projection of $\mathrm{hypo}(h)$ onto Q yields a **coherent mixed subdivision** of Q.

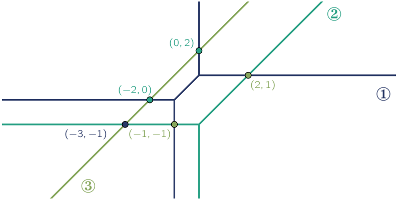
Tools for the proof of the result

Figure: The subdivision of Q associated to (E_1) arises from the projection of the Minkowski sum of the hypographs of the h_i.

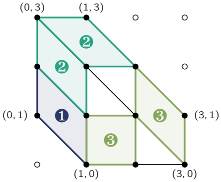


Tools for the proof of the result

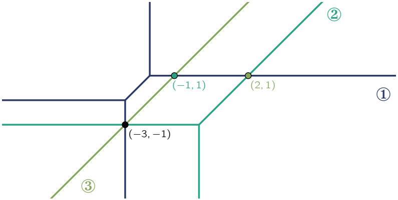
(a) The arrangement of tropical varieties of the polynomials from the system (E_1).



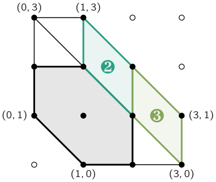
(b) The subdivision of Q associated to (E_1).



(c) The arrangement of tropical varieties of the polynomials from the system (E_2).



(d) The subdivision of Q associated to (E_2).



Nullstellensatz for Sparse Tropical Polynomial Systems

The system $f \ \nabla \ \mathbb{0}$ has a solution $x \in \mathbb{R}^n$ iff there exists a vector $y \in \mathbb{R}^{\mathcal{E}'}$ in the tropical right null space of the submatrix $\mathcal{M}_{\mathcal{E}'}$ of $\mathcal{M}$, where $\mathcal{E}'$ is any subset of $\mathbb{Z}^n$ containing a nonempty Canny-Emiris set $\mathcal{E}$.

Nullstellensatz for Sparse Tropical Polynomial Systems

The system $f \;\nabla\; \mathbb{0}$ has a solution $x \in \mathbb{R}^n$ iff there exists a vector $y \in \mathbb{R}^{\mathcal{E}'}$ in the tropical right null space of the submatrix $\mathcal{M}_{\mathcal{E}'}$ of $\mathcal{M}$, where $\mathcal{E}'$ is any subset of $\mathbb{Z}^n$ containing a nonempty Canny-Emiris set $\mathcal{E}$.

Corollary: *The system $f \;\nabla\; \mathbb{0}$ has a solution $x \in \mathbb{R}^n$ if and only if the truncated Macaulay tropical linear system $\mathcal{M}_N \odot y \;\nabla\; \mathbb{0}$ has a solution $y \in \mathbb{R}^m$ for*

$$N = d_1 + \cdots + d_k - 1 \ ,$$

where $d_i = \deg(f_i)$ for all $1 \leq i \leq k$. Moreover, if Q has full dimension, then one can take $N = d_1 + \cdots + d_k - n$ in the previous statement.

Example: The matrix associated with system (E_1) is

$$\mathcal{M}_{\mathcal{E}}^{(1)} = \begin{array}{r} \\ f_1 \\ f_2 \\ x_1 f_2 \\ x_2 f_2 \\ f_3 \\ x_1 f_3 \\ x_2 f_3 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 \end{array} \\ \begin{pmatrix} 1 & 2 & 1 & & 1 & \\ 0 & 0 & 1 & & & \\ & 0 & & 0 & 1 & \\ & & 0 & & 0 & 1 \\ & 2 & 0 & & & \\ & & & 2 & 0 & \\ & & & & 2 & 0 \end{pmatrix} \end{array}.$$

There is no finite vector in its tropical right null space and thus there is no finite solution to the equation $f \;\nabla\; \mathbb{0}$.

Example: The matrix associated with system (E_2) is

$$\mathcal{M}_{\mathcal{E}}^{(2)} = \begin{array}{r} \\ f_1 \\ f_2 \\ x_1 f_2 \\ x_2 f_2 \\ f_3 \\ x_1 f_3 \\ x_2 f_3 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & x_1 & x_2 & x_1^2 & x_1 x_2 & x_2^2 \end{array} \\ \begin{pmatrix} 1 & 4 & 1 & & 3 & \\ 0 & 0 & 1 & & & \\ & 0 & & 0 & 1 & \\ & & 0 & & 0 & 1 \\ & 2 & 0 & & & \\ & & & 2 & 0 & \\ & & & & 2 & 0 \end{pmatrix} \end{array}.$$

The vector $y = \mathrm{ver}(-3, -1) = (0, -3, -1, -6, -4, -2)$ is finite and is in the tropical right null space of the previous matrix, hence there is a finite solution to the equation $f \; \nabla \; \mathbb{0}$, which is indeed given by $(-3, -1)$.

Ingredients of the proof

A $d \times d$ tropical matrix $A = (a_{ij})_{1 \leq i,j \leq d}$ is **tropically diagonally dominant** whenever

$$a_{ii} > a_{ij}$$

for all $1 \leq i, j \leq d$ such that $i \neq j$.

Lemma: *If A is tropically diagonally dominant, then the only solution $y \in \mathbb{R}^d_\infty$ to the equation $A \odot y \ \nabla \ \mathbb{0}$ is $y = \mathbb{0}$.*

Proof: Consider $y_i = \max_{1 \leq j \leq n} y_j$, then if $y_i > -\infty$ then the inequalities $a_{ii} > a_{ij}$ and $y_i \geq y_j$ imply that

$$a_{ii} + y_i > a_{ij} + y_j \quad \text{for all} \quad 1 \leq i \neq j \leq n \ ,$$

thus contradicting the assumption that $A \odot y \ \nabla \ \mathbb{0}$.

Ingredients of the proof

- If $f \nabla \mathbb{0}$ has a solution $x \in \mathbb{R}^n$, then the **Veronese embedding** $y = \mathrm{ver}(x) := (x^p)_{p \in \mathcal{E}'}$ of x is a solution to $\mathcal{M}_{\mathcal{E}'} \odot y \nabla \mathbb{0}$.
- Otherwise we apply a construction from Canny and Emiris (1993) and Sturmfels (1994) but in a potentially **non generic** case to show that there is no finite vector $y \in \mathbb{R}^{\mathcal{E}'}$ in the tropical right null space of $\mathcal{M}_{\mathcal{E}'}$.

Ingredients of the proof

- If $p \in \mathcal{E}$, then $(p-\delta, h(p-\delta))$ is in the **relative interior** of a facet F of hypo(h), and F can be written as $F_1 + \cdots + F_k$ with F_i faces of hypo(h_i).
- Since f does not have a common root, at least one F_i is a singleton. Consider the maximal index j such that $F_j = \{a_j\}$ is a **singleton**. The couple (j, a_j) is called the **row content** of p.
- If $p \in \mathcal{E}$ and if (j, a_j) is its row content, then the support of the polynomial $X^{p-a_j} f_j$ is included in $\mathcal{E}$. This allows us to construct a square submatrix $\mathcal{M}_{\mathcal{E}\mathcal{E}} = (m_{pp'})_{(p,p') \in \mathcal{E} \times \mathcal{E}}$ of $\mathcal{M}_{\mathcal{E}}$.

Ingredients of the proof

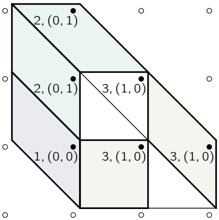
- The matrix $\widetilde{\mathcal{M}}_{\mathcal{E}\mathcal{E}} = (\widetilde{m}_{pp'})_{(p,p')\in\mathcal{E}\times\mathcal{E}}$ obtained by setting $\widetilde{m}_{pp'} = m_{pp'} - h(p' - \delta)$ is **tropically diagonally dominant**.
- Therefore its tropical right null space is reduced to $\{\mathbb{0}\}$, and thus this is also the case for $\mathcal{M}_{\mathcal{E}\mathcal{E}}$.
- Hence there does not exist $y \in \mathbb{R}^{\mathcal{E}}$ such that $\mathcal{M}_{\mathcal{E}} \odot y \; \nabla \; \mathbb{0}$.
- Finally, since $\mathcal{M}_{\mathcal{E}'}$ can be written by block as

$$\mathcal{M}_{\mathcal{E}'} = \begin{array}{c} \\ \end{array}\begin{matrix} \mathcal{E} & \mathcal{E}' \setminus \mathcal{E} \\ \end{matrix} \atop \begin{pmatrix} \mathcal{M}_{\mathcal{E}} & \mathbb{0} \\ * & * \end{pmatrix},$$

we deduce that there does also not exist $y \in \mathbb{R}^{\mathcal{E}'}$ such that $\mathcal{M}_{\mathcal{E}'} \odot y \; \nabla \; \mathbb{0}$.

Ingredients of the proof

Figure: The polytope $Q + \delta$, with the integer points inside the maximal dimensional cells of the decomposition of $Q + \delta$ labelled by the row content the cell they belong to.



This configuration yields the following nonsingular square submatrix of $\mathcal{M}_{\mathcal{E}}^{(1)}$

$$\mathcal{M}_{\mathcal{E}\mathcal{E}}^{(1)} = \begin{array}{r} \\ (0,0) \to f_1 \\ (1,0) \to f_3 \\ (0,1) \to f_2 \\ (2,0) \to x_1 f_3 \\ (1,1) \to x_2 f_3 \\ (0,2) \to x_2 f_2 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & x_1 & x_2 & x_1^2 & x_1 x_2 & x_2^2 \end{array} \\ \begin{pmatrix} 1 & 2 & 1 & & 1 & \\ & 2 & 0 & & & \\ 0 & 0 & 1 & & & \\ & & & 2 & 0 & \\ & & & & 2 & 0 \\ & & 0 & & 0 & 1 \end{pmatrix} \end{array}.$$

IV - The tropical Positivstellensatz for sparse polynomial systems

- Let $f^{\pm} = (f_1^{\pm}, \dots, f_k^{\pm})$ be two collections of tropical polynomials. For $1 \leq i \leq k$, denote by $\mathcal{A}_i^{\pm}$ the support of $f_i^{\pm}$ and let $f_i = f_i^{+} \oplus f_i^{-}$, with support $\mathcal{A}_i = \mathcal{A}_i^{+} \cup \mathcal{A}_i^{-}$.

- Let $f^{\pm} = (f_1^{\pm}, \dots, f_k^{\pm})$ be two collections of tropical polynomials. For $1 \leq i \leq k$, denote by $\mathcal{A}_i^{\pm}$ the support of $f_i^{\pm}$ and let $f_i = f_i^{+} \oplus f_i^{-}$, with support $\mathcal{A}_i = \mathcal{A}_i^{+} \cup \mathcal{A}_i^{-}$.
- Set $\rhd = (\rhd_1, \dots, \rhd_k)$ a collection of relations, with $\rhd_i \in \{\geq, =, >\}$ for $1 \leq i \leq k$.

We denote by $f^{+}(x) \rhd f^{-}(x)$ the system

$$\max_{\alpha \in \mathcal{A}_i^{+}} \left(f_{i,\alpha}^{+} + \langle \alpha, x \rangle\right) \rhd_i \max_{\alpha \in \mathcal{A}_{i^-}} \left(f_{i,\alpha}^{-} + \langle \alpha, x \rangle\right) \text{ for all } 1 \leq i \leq k$$

of unknown $x \in \mathbb{R}_{\infty}^{n}$.

- Let $\mathcal{M}^{\pm}$ be the Macaulay matrices associated to $f^{\pm}$ – *i.e.* with entries $f^{\pm}_{i,\beta-\alpha}$. For any subset $\mathcal{E}$ of $\mathbb{Z}^n$, denote by $\mathcal{M}^{\pm}_{\mathcal{E}}$ the submatrices of $\mathcal{M}^{\pm}$ by taking only the row indices $(i,\alpha) \in [k] \times \mathbb{Z}^n$ such that the supports of the rows (i,α) of both $\mathcal{M}^{+}$ and $\mathcal{M}^{-}$ is included in $\mathcal{E}$ and the column indices given by $\mathcal{E}$.

- Let $\mathcal{M}^{\pm}$ be the Macaulay matrices associated to $f^{\pm}$ – *i.e.* with entries $f^{\pm}_{i,\beta-\alpha}$. For any subset $\mathcal{E}$ of $\mathbb{Z}^n$, denote by $\mathcal{M}^{\pm}_{\mathcal{E}}$ the submatrices of $\mathcal{M}^{\pm}$ by taking only the row indices $(i,\alpha) \in [k] \times \mathbb{Z}^n$ such that the supports of the rows (i,α) of both $\mathcal{M}^{+}$ and $\mathcal{M}^{-}$ is included in $\mathcal{E}$ and the column indices given by $\mathcal{E}$.
- Finally, denote by $\mathcal{M}^{+}_{\mathcal{E}} \odot y \rhd \mathcal{M}^{-}_{\mathcal{E}} \odot y$ the following system of tropical linear inequalities:

$$\max_{\beta\in\mathcal{E}} \left(\mathcal{M}^{+}_{(i,\alpha),\beta} + y_\beta\right) \rhd_i \max_{\beta\in\mathcal{E}} \left(\mathcal{M}^{-}_{(i,\alpha),\beta} + y_\beta\right) \text{ for all } 1 \leq i \leq k.$$

Let $\widetilde{Q} = r_1 Q_1 + \cdots + r_k Q_k$, where $Q_i = \operatorname{conv}(\mathcal{A}_i)$ for $i = 1, \ldots, k$, and

$$r_i = \begin{cases} \min(|\mathcal{A}_i^-|, n+1) & \text{if } \rhd_i \in \{\geq, >\} \\ \min(\max(|\mathcal{A}_i^-|, |\mathcal{A}_i^+|), n+1) & \text{if } \rhd_I \in \{=\} \end{cases} .$$

Let $\widetilde{Q} = r_1 Q_1 + \cdots + r_k Q_k$, where $Q_i = \mathrm{conv}(\mathcal{A}_i)$ for $i = 1, \ldots, k$, and

$$r_i = \begin{cases} \min(|\mathcal{A}_i^-|, n+1) & \text{if } \triangleright_i \in \{\geq, >\} \\ \min(\max(|\mathcal{A}_i^-|, |\mathcal{A}_i^+|), n+1) & \text{if } \triangleright_I \in \{=\} \end{cases} .$$

We now call **Canny-Emiris subsets** of $\mathbb{Z}^n$ associated to the pair of collections (f^+, f^-) any set $\mathcal{E}$ of the form

$$\mathcal{E} := \left(\widetilde{Q} + \delta\right) \cap \mathbb{Z}^n \quad ,$$

where δ is a generic vector in $V + \mathbb{Z}^n$, with V the direction of the affine hull of $\widetilde{Q}$.

Main ingredient of the proof

The Shapley-Folkman Lemma

Let $A_1, \ldots, A_k \subseteq \mathbb{R}^n$, and let

$$x \in \sum_{i=1}^{k} \operatorname{conv}(A_i) \ .$$

Then there is an index set $I \subseteq \{1, \ldots, k\}$ with $|I| \leq n$ such that

$$x \in \sum_{i \in I} \operatorname{conv}(A_i) + \sum_{i \in \{1,\ldots,k\} \setminus I} A_i \ .$$

Corollary: If $\sum_{i=1}^{k} \operatorname{conv}(A_i)$ has (affine) dimension $d < n$, then the index set I can be choosen such that $|I| \leq d$.

Tropical Positivstellensatz

There exists a solution $x \in \mathbb{R}^n$ to the system $f^+(x) \rhd f^-(x)$ if and only if there exists a vector $y \in \mathbb{R}^{\mathcal{E}'}$ satisfying $\mathcal{M}^+_{\mathcal{E}'} \odot y \rhd \mathcal{M}^-_{\mathcal{E}'} \odot y$, where $\mathcal{E}'$ is any subset of $\mathbb{Z}^n$ containing a nonempty Canny-Emiris subset $\mathcal{E}$ of $\mathbb{Z}^n$ associated to the pair (f^+, f^-).

Corollary: Let $f_0^\pm, \dots, f_k^\pm$ be a collection of pairs of tropical polynomials. Then, the following implication holds for all $x \in \mathbb{R}^n$

$$\left(\forall 1 \le i \le k,\ f_i^+(x) \ge f_i^-(x)\right) \quad \implies \quad f_0^+(x) \ge f_0^-(x)$$

iff the Macaulay linearization $\mathcal{M}^+_{\mathcal{E}'} \odot y \rhd \mathcal{M}^-_{\mathcal{E}'} \odot y$ associated to the relations $f_i^+(x) \ge f_i^-(x)$ for $i = 1, \dots, k$ and $f_0^+(x) < f_0^-(x)$, where $\mathcal{E}'$ is as above, has no finite solution y.

V - Algorithmical aspects

Mean payoff games (See **Gillette (1957)**,
Gurvich, Karzanov, Khachiyan (1988),
Zwick, Patterson (1996)):

Mean payoff games (See **Gillette (1957)**, **Gurvich, Karzanov, Khachiyan (1988)**, **Zwick, Patterson (1996)**):

- $G = (I \sqcup J, E)$ a (finite) oriented weighter bipartite graph;

Mean payoff games (See **Gillette (1957)**,
Gurvich, Karzanov, Khachiyan (1988),
Zwick, Patterson (1996)):

- $G = (I \sqcup J, E)$ a (finite) oriented weighter bipartite graph;
- game with two players Min and Max:

Mean payoff games (See **Gillette (1957)**,
Gurvich, Karzanov, Khachiyan (1988),
Zwick, Patterson (1996)):

- $G = (I \sqcup J, E)$ a (finite) oriented weighter bipartite graph;
- game with two players Min and Max: each turn, from the current state $i \in I$, player Max chooses a state $j \in J$ such that (i, j) is an arc of G with weight b_{ij} and obtains a payment of b_{ij} from player Min,

Mean payoff games (See **Gillette (1957)**,
Gurvich, Karzanov, Khachiyan (1988),
Zwick, Patterson (1996)):

- $G = (I \sqcup J, E)$ a (finite) oriented weighter bipartite graph;
- game with two players Min and Max: each turn, from the current state $i \in I$, player Max chooses a state $j \in J$ such that (i, j) is an arc of G with weight b_{ij} and obtains a payment of b_{ij} from player Min, then player Min from state $j \in J$, chooses the next state $k \in I$ along an arc (k, j) with weight a_{kj}, and receives in turn a payment of $a_{kj}$from player Max ;

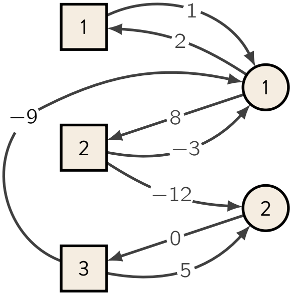
Mean payoff games (See **Gillette (1957)**,
Gurvich, Karzanov, Khachiyan (1988),
Zwick, Patterson (1996)):

- $G = (I \sqcup J, E)$ a (finite) oriented weighter bipartite graph;
- game with two players Min and Max: each turn, from the current state $i \in I$, player Max chooses a state $j \in J$ such that (i, j) is an arc of G with weight b_{ij} and obtains a payment of b_{ij} from player Min, then player Min from state $j \in J$, chooses the next state $k \in I$ along an arc (k, j) with weight a_{kj}, and receives in turn a payment of $a_{kj}$from player Max ;
- the winner is the player who gets the highest average payment per turn;

Mean payoff games (See **Gillette (1957)**, **Gurvich, Karzanov, Khachiyan (1988)**, **Zwick, Patterson (1996)**):

- $G = (I \sqcup J, E)$ a (finite) oriented weighter bipartite graph;
- game with two players Min and Max: each turn, from the current state $i \in I$, player Max chooses a state $j \in J$ such that (i, j) is an arc of G with weight b_{ij} and obtains a payment of b_{ij} from player Min, then player Min from state $j \in J$, chooses the next state $k \in I$ along an arc (k, j) with weight a_{kj}, and receives in turn a payment of $a_{kj}$from player Max ;
- the winner is the player who gets the highest average payment per turn;
- set $A = (a_{ij})_{(i,j) \in I \times J}$ et $B = (b_{ij})_{(i,j) \in I \times J}$.

Example : Let G be the following graph:

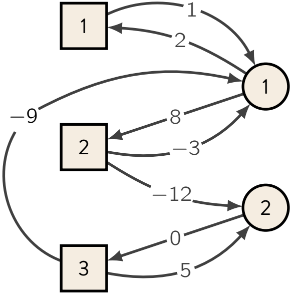


$$\text{One has } A = \begin{pmatrix} 2 & -\infty \\ 8 & -\infty \\ -\infty & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & -\infty \\ -3 & -12 \\ -9 & 5 \end{pmatrix}.$$

Theorem [Akian, Gaubert, Guterman (2012)] : For all $j \in J$, player Max has a winning positional strategy for the *mean pay-off game* given by the payment matrices A and B by playing the initial move j iff there exists a solution $y \in \mathbb{R}_{\infty}^{J}$ of the tropical matrix inequality $A \odot y \leq B \odot y$ such that $y_j \neq \mathbb{0}$.

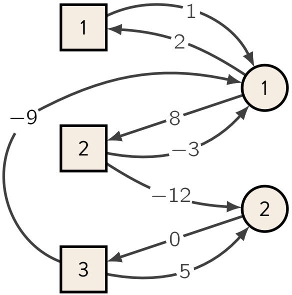
The winning initial moves correspond to the support of the solutions of the inequality $A \odot y \leq B \odot y$.

In the previous example,



one has $A \odot y \leq B \odot y \iff \begin{cases} 2 + y_1 & \leq & 1 + y_1 \\ 8 + y_1 & \leq & \max(-3 + y_1, -12 + y_2) \\ y_2 & \leq & \max(-9 + y_1, 5 + y_2). \end{cases}$

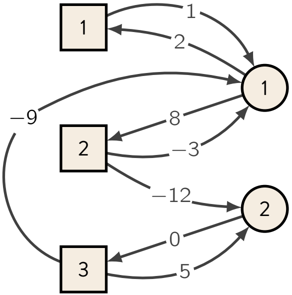
In the previous example,



one has $A \odot y \leq B \odot y \iff \begin{cases} 2 + y_1 \leq 1 + y_1 \\ 8 + y_1 \leq \max(-3 + y_1, -12 + y_2) \\ y_2 \leq \max(-9 + y_1, 5 + y_2). \end{cases}$

The first inequality shows that every solution $y \in \mathbb{R}_\infty^2$ must satisfy $y_1 = \mathbb{0}$, which implies that the two other inesualities are satisfied for all values of $y_2 \in \mathbb{R}_\infty$.

In the previous example,



one has $A \odot y \leq B \odot y \iff \begin{cases} 2 + y_1 & \leq & 1 + y_1 \\ 8 + y_1 & \leq & \max(-3 + y_1, -12 + y_2) \\ y_2 & \leq & \max(-9 + y_1, 5 + y_2). \end{cases}$

The first inequality shows that every solution $y \in \mathbb{R}^2_\infty$ must satisfy $y_1 = \mathbb{0}$, which implies that the two other inesualities are satisfied for all values of $y_2 \in \mathbb{R}_\infty$.

This translates into the fact that the move 1 is a losing move for player Max, while the move 2 is a winning move.

- **Shapley operator** associated to a mean payoff game

$$T: \begin{array}{ccc} \mathbb{R} \cup \{\pm\infty\} & \longrightarrow & \mathbb{R} \cup \{\pm\infty\} \\ y = (y_j)_{j \in J} & \mapsto & \left(\min_{i \in I} -a_{ik} + \left(\max_{j \in J} b_{ij} + y_j\right)\right)_{k \in J} \end{array}$$

- **Shapley operator** associated to a mean payoff game

$$T: \begin{array}{ccc} \mathbb{R} \cup \{\pm\infty\} & \longrightarrow & \mathbb{R} \cup \{\pm\infty\} \\ y = (y_j)_{j \in J} & \mapsto & \left(\min_{i \in I} -a_{ik} + \left(\max_{j \in J} b_{ij} + y_j\right)\right)_{k \in J} \end{array}$$

- **value** of the game: $\chi(T) = \lim_{n \to +\infty} \frac{T^n(0)}{n}$

- **Shapley operator** associated to a mean payoff game

$$T: \begin{array}{ccc} \mathbb{R} \cup \{\pm\infty\} & \longrightarrow & \mathbb{R} \cup \{\pm\infty\} \\ y = (y_j)_{j \in J} & \mapsto & \left(\min_{i \in I} -a_{ik} + \left(\max_{j \in J} b_{ij} + y_j\right)\right)_{k \in J} \end{array}$$

- **value** of the game: $\chi(T) = \lim_{n \to +\infty} \frac{T^n(0)}{n}$

Corollary: $\exists y \in \mathbb{R}^n$ such that $A \odot y \leq B \odot y$ iff $\chi(f) \geq 0$.

- **Shapley operator** associated to a mean payoff game

$$T: \begin{array}{ccc} \mathbb{R} \cup \{\pm\infty\} & \longrightarrow & \mathbb{R} \cup \{\pm\infty\} \\ y = (y_j)_{j \in J} & \mapsto & \left(\min_{i \in I} -a_{ik} + \left(\max_{j \in J} b_{ij} + y_j\right)\right)_{k \in J} \end{array}$$

- **value** of the game: $\chi(T) = \lim_{n \to +\infty} \frac{T^n(0)}{n}$

Corollary: $\exists y \in \mathbb{R}^n$ such that $A \odot y \leq B \odot y$ iff $\chi(f) \geq 0$.

- The existence of a polynomial time algorithm to solve mean payoff games is an open problem since 1988, but but there exist practically fast methods (value/policy iteration algorithms).

Value iteration algorithm

Algorithm 1: Value iteration algorithm with widening.

input: T a Shapley operator from $(\mathbb{R} \cup \{+\infty\})^m$ to $(\mathbb{R} \cup \{+\infty\})^m$ $\varepsilon > 0$ the approximation error for comparisons

output: Decides the feasibility of the system $A \odot y \leq B \odot y$ in $\mathbb{R}^m$

procedure ValueIteration(T, ε):
$u := 0 \in \mathbb{R}^m$
$v := 0 \in \mathbb{R}^m$
repeat
  /* value iteration step */
  $u := v$
  $v := u \wedge T(u)$
  /* widening step */
  $I := \{i : v_i \geq -\varepsilon + u_i\}$
  $\tilde{u} := (\tilde{u}_i) \in (\mathbb{R} \cup \{+\infty\})^m$ with $\tilde{u}_i = +\infty$ if $i \in I$ and $\tilde{u}_i = u_i$ otherwise
  $\tilde{v} := T(\tilde{u})$
until $v \geq -\varepsilon + u$ or $v \ll -\varepsilon + u$ or $\tilde{v} \ll -\varepsilon + \tilde{u}$
if $v \ll -\varepsilon + u$ or $\tilde{v} \ll -\varepsilon + \tilde{u}$ **then**
  /* No finite vector y satisfies $T(y) \geq -\varepsilon + y$. */
  return "Unfeasible"
else
  /* The vector u satisfies $T(u) \geq -\varepsilon + u$. */
  return "Feasible"

For two vectors $u, v \in (\mathbb{R} \cup \{+\infty\})^n$, we write $v \ll u$ if for all i such that $u_i < +\infty$, we have $v_i < u_i$, and for $\lambda \in \mathbb{R}$, we denote $\lambda + u$ the vector with coordinates $\lambda + u_i$.

Notice that the time of a single iteration is proportional to the number of nonzero entries of the matrix.

Python implementation of the algorithm available at:

`https://gitlab.inria.fr/abereau/tropical-polynomial-system-solving`

Python implementation of the algorithm available at:

`https://gitlab.inria.fr/abereau/tropical-polynomial-system-solving`

The bottleneck resides mainly in the computation of the Minkowski sum of the Newton polytopes of the polynomials of the system.

Python implementation of the algorithm available at:

`https://gitlab.inria.fr/abereau/tropical-polynomial-system-solving`

The bottleneck resides mainly in the computation of the Minkowski sum of the Newton polytopes of the polynomials of the system.

Table: Average number of columns in the Macaulay matrices in the sparse case (right) for random systems of k inequations in n variables among 100 samples, compared to the number of columns in the full case (left).

		k														
		2			3			4			5			6		
n	2	45	-	35	91	-	73	153	-	128	231	-	193	325	-	276
	3	165	-	85	455	-	265	969	-	611	1771	-	1156	2925	-	1987
	4	495	-	138	1820	-	651	4845	-	2079	10626	-	5044	20475	-	10418
	5	1287	-	163	6188	-	1268	20349	-	5165	53130	-	$\times$	118755	-	$\times$
	6	3003	-	154	18564	-	$\sim$1300	74613	-	$\times$	230230	-	$\times$	593775	-	$\times$

Table: Number of feasible instances for random systems of k inequations in n variables among 100 samples.

		k								
		2	3	4	5	6	7	8	9	10
n	2	97	91	62	48	38	28	12	12	5
	3	100	98	97	79	74	62	48	32	17
	4	100	100	100	100	93	92	80	×	×
	5	100	100	100	×	×	×	×	×	×
	6	100	100	×	×	×	×	×	×	×

Table: Average runtime in seconds to solve an instance of a random system of k inequations in n variables among 100 samples.

		k								
		2	3	4	5	6	7	8	9	10
n	2	0.04	0.16	0.67	1.58	2.91	3.73	2.43	4.55	2.53
	3	0.08	0.71	3.82	8.80	33.45	84.87	183.26	180.58	154.43
	4	0.74	2.95	11.54	48.47	266.95	654.06	1952.40	×	×
	5	5.09	64.94	312.66	×	×	×	×	×	×
	6	67.60	2427.67	×	×	×	×	×	×	×

Incoming work:

- Faster algorithm using the policy iteration method
- Tropical analog of eigenvalue methods: we can effectively recover the solution by solving a parametric mean payoff game (current work)
- Open problem: can the degree bound be improved in the Positivstellensatz (no tight example found yet)

Thank you for your attention!