Eigenvalue Methods for Sparse Tropical Polynomial Systems

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- Given system of tropical polynomial equations or inequations, how to check the existence of, and then compute a solution in $\mathbb{R}^n$.
- Main tools in the classical setting include the theory of resultants, Macaulay matrices and effective Null- and Positivstellensatz.
- In this talk, we develop the tropical analog of the sparse Null- and Positivstellensatz, and explore the solvability of tropical polynomial systems by means of mean payoff games.

1 **Tropical algebra and tropical polynomials**

2 **The tropical Null- and Positivstellensatz**

3 **Mean payoff games and tropical linear systems**

I - Tropical algebra and tropical polynomials

- **Tropical semiring** $\mathbb{T} := (\mathbb{R} \cup \{-\infty\}, \oplus, \odot, \mathbb{0}, \mathbb{1})$ with
  - addition $\oplus := \max$;
  - multiplication $\odot := +$;
  - zero element $\mathbb{0} := -\infty$;
  - unit element $\mathbb{1} := 0$.
- Satisfies the usual properties of a field except **no additive inverse**.
- Tropical operations can be extended to vectors and matrices with coefficients in $\mathbb{R} \cup \{-\infty\}$ to perform **tropical linear algebra**.

- A **formal tropical polynomial** p in n variables is a map

$$\begin{array}{ccc} \mathbb{Z}^n & \longrightarrow & \mathbb{R} \cup \{-\infty\} \\ \alpha & \longmapsto & p_\alpha \end{array}$$

  such that $p_\alpha \neq \mathbb{0}$ for finitely many $\alpha \in \mathbb{Z}^n$. We denote $p = \bigoplus_{\alpha \in \mathbb{Z}^n} p_\alpha X^\alpha$.
- **Support** of p: $\mathrm{supp}(p) := \{\alpha \in \mathbb{Z}^n : p_\alpha \neq \mathbb{0}\}$.
- **Polynomial function** associated to p:

$$\hat{p} : \left\{ \begin{array}{ccc} \mathbb{R}^n & \longrightarrow & \mathbb{R} \cup \{-\infty\} \\ x & \longmapsto & \hat{p}(x) := \max_{\alpha \in \mathcal{A}}(p_\alpha + \langle x, \alpha \rangle) \end{array} \right.$$

  with $\mathcal{A} = \mathrm{supp}(p)$.

A point $x \in (\mathbb{R} \cup \{-\infty\})^n$ is a **root** of a polynomial p whenever the maximum in the expression

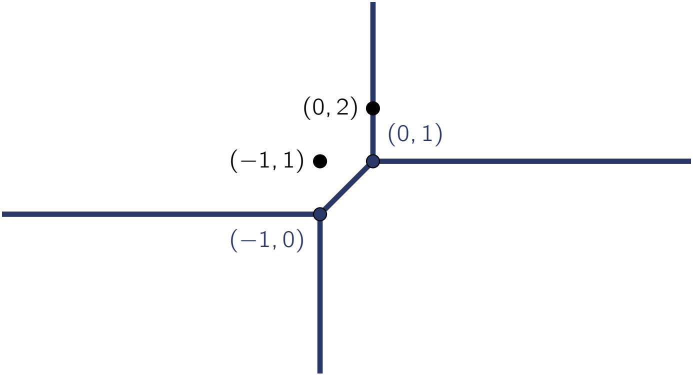
$$\hat{p}(x) = \bigoplus_{\alpha \in \mathcal{A}} p_\alpha \odot x^{\odot \alpha} = \max_{\alpha \in \mathcal{A}} (p_\alpha + \langle x, \alpha \rangle)$$

is attained for at least two distinct values of α. This is denoted as $p(x) \mathrel{\nabla} \mathbb{0}$.

Exemple : Let $f_1 = 1 \oplus 2x_1 \oplus 1x_2 \oplus 1x_1x_2$, then:

- $(0, 2)$ is a root of f_1 since the maximum of $\hat{f}_1(0, 2) = 3$ is attained simultaneously by the monomials $1x_2$ and $1x_1x_2$;
- $(-1, 1)$ is not a root of f_1 since the maximum $\hat{f}_1(-1, 1) = 2$ is attained only by the monomial $1x_2$.

The tropical hypersurface associated to the polynomial f_1 is:



Likewise, $y \in (\mathbb{R} \cup \{-\infty\})^m$ is said to be in the **tropical right null space** or **kernel** of a $\ell \times m$ matrix $A = (a_{ij})$ whenever for all $1 \leq i \leq \ell$, the maximum in the expression

$$\bigoplus_{j=1}^{m} a_{ij} \odot y_j = \max_{1 \leq j \leq m} (a_{ij} + y_j)$$

is achieved at least twice. This is also denoted as $A \odot y \; \nabla \; \mathbb{0}$.

More on tropical geometry: D. Maclagan and B. Sturmfels. *Introduction to Tropical Geometry*. Graduate Studies in Mathematics. American Mathematical Society, 2015.

II - The tropical Null- and Positivstellensatz

In the following, we fix a collection $f = (f_1, \ldots, f_k)$ of k formal tropical polynomials in n variables, with respective supports $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_k)$ and degrees $(d_1, \ldots, d_k)$.

Problem: Decide whether there is a common tropical zero $x \in \mathbb{R}^n$, that is such that $f_i(x) \nabla \mathbb{0}$ for all $1 \leq i \leq n$.

Remark: The same question for a solution in $(\mathbb{R} \cup \{-\infty\})^n$ reduces to the $\mathbb{R}^n$ case by looking at all possible supports.

Figure: The arrangement of tropical varieties of the polynomials from the system

$$(E_1):\begin{cases} f_1 = 1 \oplus 2x_1 \oplus 1x_2 \oplus 1x_1x_2 \\ f_2 = 0 \oplus 0x_1 \oplus 1x_2 \\ f_3 = 2x_1 \oplus 0x_2 \ , \end{cases}.$$

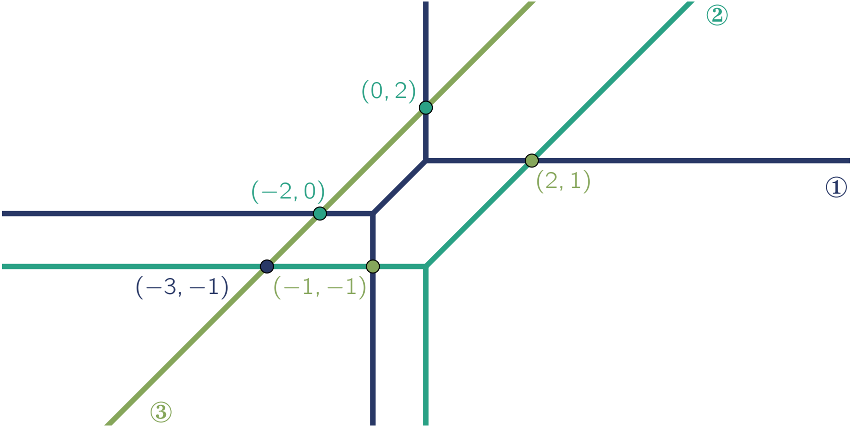
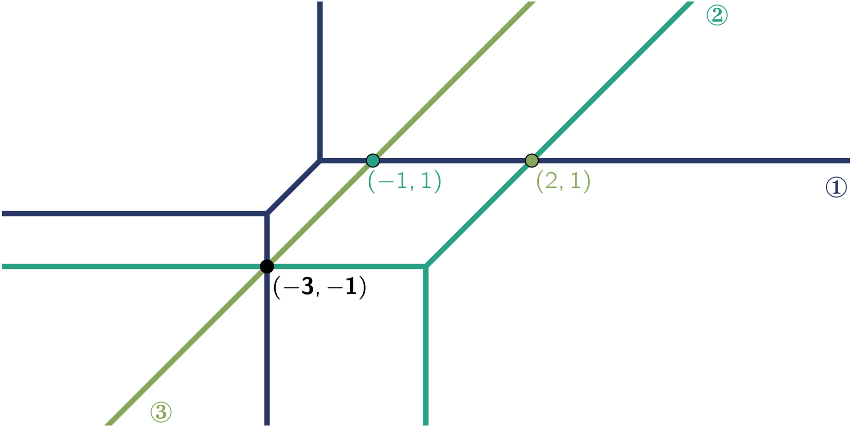


Figure: The arrangement of tropical varieties of the polynomials from the system

$$(E_2):\begin{cases} f_1 &= 1 \oplus 4x_1 \oplus 1x_2 \oplus 3x_1x_2 \\ f_2 &= 0 \oplus 0x_1 \oplus 1x_2 \\ f_3 &= 2x_1 \oplus 0x_2 \;, \end{cases} .$$



Link with classical varieties:

- Kapranov's theorem
- The Fundamental Theorem of Tropical Algebraic Geometry

Varied applications:

- celestial mechanics (Hampton, Moeckel)
- max-out networks (Montúfar, Ren, Zhang)
- chemical reaction networks (Dickenstein, Feliu, Radulescu, Shiu)
- emergency call center (Akian, Boyer, Gaubert)

The **Macaulay matrix** associated to f is the infinite matrix $\mathcal{M} = (m_{(i,\alpha),\beta})$ indexed by $([n] \times \mathbb{Z}^n) \times \mathbb{Z}^n$, where $m_{(i,\alpha),\beta}$ corresponds to the coefficient of X^β in the polynomial $X^\alpha f_i$.

$$\mathcal{M} = \begin{array}{c} \\ f_1 \\ x_1 f_1 \\ \vdots \\ x^\alpha f_i \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & x_1 & \cdots & x^\beta & \cdots \end{array} \\ \begin{pmatrix} * & * & \cdots & * & \cdots \\ * & * & \cdots & * & \cdots \\ \vdots & \vdots & \ddots & \vdots & \\ * & * & \cdots & f_{i,\beta-\alpha} & \cdots \\ \vdots & \vdots & & \vdots & \ddots \end{pmatrix} \end{array}$$

One approch to the **Nullstellensatz** is a linearization method reducing the search of a solution of a polynomial system to the search of nonzero elements in the kernel of the Macaulay matrix.

- The Macaulay matrix associated to f is the **infinite** matrix $\mathcal{M} = (m_{(i,\alpha),\beta})$ indexed by $([n] \times \mathbb{Z}^n) \times \mathbb{Z}^n$, where $m_{(i,\alpha),\beta}$ corresponds to the coefficient of X^β in the polynomial $X^\alpha f_i$.
- A finite subset $\mathcal{E}$ of $\mathbb{Z}^n$ yields a **finite** submatrix $\mathcal{M}_\mathcal{E}$ of $\mathcal{M}$ obtained by taking only the rows whose support is included in $\mathcal{E}$ and the columns indexed by $\mathcal{E}$.
- Set $\mathcal{M}_N := \mathcal{M}_\mathcal{E}$ for $\mathcal{E} = \{\alpha \in \mathbb{N}^n : \alpha_1 + \cdots + \alpha_n \leq N\}$.

Conjecture [Grigoriev (2012)]: There exists an integer N such that

$$\exists x \in \mathbb{R}^n \text{ such that } f_i(x) \nabla \mathbb{0} \text{ for } i = 1, \ldots, k$$

$$\Longleftrightarrow$$

$$\exists y \in \mathbb{R}^m \text{ such that } \mathcal{M}_N \odot y \nabla \mathbb{0} \text{ with } m = \binom{N+n}{n} \ .$$

Answer:

- **Grigoriev, Podolskii (2018):** true for

$$N = (n+2)(d_1 + \cdots + d_k) \ .$$

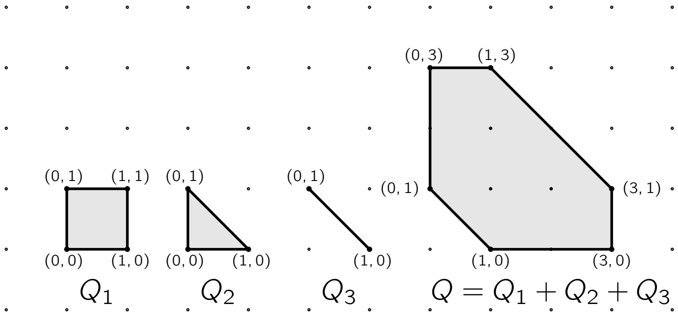
- **Akian, B., Gaubert (2023):** true for

$$N = d_1 + \cdots + d_k - 1$$

(and even $N = d_1 + \cdots + d_k - n$ in most cases) + adapted approch for the case of sparse polynomials.

- For $1 \leq i \leq k$, $Q_i := \mathrm{conv}(\mathcal{A}_i)$ is the **Newton polytope** of f_i.

Example: The Newton polytopes associated to both system (E_1) and system (E_2) and their Minkowski sum are as follow.



- **Canny-Emiris set** associated to f: $\mathcal{E} = (Q + \delta) \cap \mathbb{Z}^n$ with δ a generic vector in the linear space directing the affine hull of Q.

Example: Considering again the systems (E_1) and (E_2), for

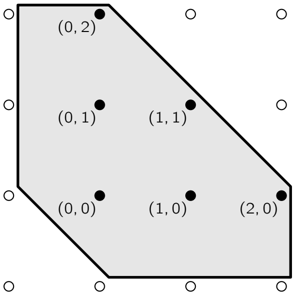
$$\delta = (-1 + \varepsilon, -1 + \varepsilon)$$

with $\varepsilon > 0$ sufficiently small, we obtain the Canny-Emiris set

$$\mathcal{E} := (Q + \delta) \cap \mathbb{Z}^n = \{(0,0), (1,0), (0,1), (2,0), (1,1), (0,2)\}$$

corresponding to the set of monomials $\{1, x_1, x_2, x_1^2, x_1x_2, x_2^2\}$.

Figure: The polytope $Q + \delta$ with $\delta = (-0.9, -0.9)$.



Nullstellensatz for Sparse Tropical Polynomial Systems

The system $f \nabla \mathbb{0}$ has a solution $x \in \mathbb{R}^n$ iff there exists a vector $y \in \mathbb{R}^{\mathcal{E}'}$ in the tropical right null space of the submatrix $\mathcal{M}_{\mathcal{E}'}$ of $\mathcal{M}$, where $\mathcal{E}'$ is any subset of $\mathbb{Z}^n$ containing a nonempty Canny-Emiris set $\mathcal{E}$.

Corollary: *The system $f \nabla \mathbb{0}$ has a solution $x \in \mathbb{R}^n$ if and only if the truncated Macaulay tropical linear system $\mathcal{M}_N \odot y \nabla \mathbb{0}$ has a solution $y \in \mathbb{R}^m$ for*

$$N = d_1 + \cdots + d_k - 1 \ ,$$

where $d_i = \deg(f_i)$ for all $1 \leq i \leq k$. Moreover, if Q has full dimension, then one can take $N = d_1 + \cdots + d_k - n$ in the previous statement.

Example: The matrix associated with system (E_1) is

$$\mathcal{M}_{\mathcal{E}}^{(1)} = \begin{array}{r} \\ f_1 \\ f_2 \\ x_1 f_2 \\ x_2 f_2 \\ f_3 \\ x_1 f_3 \\ x_2 f_3 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 \end{array} \\ \begin{pmatrix} 1 & 2 & 1 & & 1 & \\ 0 & 0 & 1 & & & \\ & 0 & & 0 & 1 & \\ & & 0 & & 0 & 1 \\ & 2 & 0 & & & \\ & & & 2 & 0 & \\ & & & & 2 & 0 \end{pmatrix} \end{array} .$$

There is no finite vector in its tropical right null space and thus there is no finite solution to the equation $f \ \nabla \ \mathbb{0}$.

Example: The matrix associated with system (E_2) is

$$\mathcal{M}_{\mathcal{E}}^{(2)} = \begin{array}{c} \\ f_1 \\ f_2 \\ x_1 f_2 \\ x_2 f_2 \\ f_3 \\ x_1 f_3 \\ x_2 f_3 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & x_1 & x_2 & x_1^2 & x_1 x_2 & x_2^2 \end{array} \\ \begin{pmatrix} 1 & 4 & 1 & & 3 & \\ 0 & 0 & 1 & & & \\ & 0 & & 0 & 1 & \\ & & 0 & & 0 & 1 \\ & 2 & 0 & & & \\ & & & 2 & 0 & \\ & & & & 2 & 0 \end{pmatrix} \end{array} .$$

The vector $y = \mathrm{ver}(-3, -1) = (0, -3, -1, -6, -4, -2)$ is finite and is in the tropical right null space of the previous matrix, hence there is a finite solution to the equation $f \ \nabla \ \mathbb{0}$, which is indeed given by $(-3, -1)$.

- Let $f^{\pm} = (f_1^{\pm}, \dots, f_k^{\pm})$ be two collections of tropical polynomials. For $1 \leq i \leq k$, denote by $\mathcal{A}_i^{\pm}$ the support of $f_i^{\pm}$.
- Set $\rhd = (\rhd_1, \dots, \rhd_k)$ a collection of relations, with $\rhd_i \in \{\geq, =, >\}$ for $1 \leq i \leq k$.

We denote by $f^+(x) \rhd f^-(x)$ the system

$$\max_{\alpha \in \mathcal{A}_i^+} \left(f_{i,\alpha}^+ + \langle \alpha, x \rangle\right) \rhd_i \max_{\alpha \in \mathcal{A}_{i^-}} \left(f_{i,\alpha}^- + \langle \alpha, x \rangle\right) \text{ for all } 1 \leq i \leq k$$

of unknown $x \in (\mathbb{R} \cup \{-\infty\})^n$.

- Let $\mathcal{M}^{\pm}$ be the Macaulay matrices associated to $f^{\pm}$ – *i.e.* with entries $f^{\pm}_{i,\beta-\alpha}$. For any subset $\mathcal{E}$ of $\mathbb{Z}^n$, denote by $\mathcal{M}^{\pm}_{\mathcal{E}}$ the submatrices of $\mathcal{M}^{\pm}$ by taking only the row indices $(i,\alpha) \in [k] \times \mathbb{Z}^n$ such that the supports of the rows (i,α) of both $\mathcal{M}^+$ and $\mathcal{M}^-$ is included in $\mathcal{E}$ and the column indices given by $\mathcal{E}$.
- Finally, denote by $\mathcal{M}^+_{\mathcal{E}} \odot y \rhd \mathcal{M}^-_{\mathcal{E}} \odot y$ the following system of tropical linear inequalities:

$$\max_{\beta\in\mathcal{E}} \left(\mathcal{M}^+_{(i,\alpha),\beta} + y_\beta\right) \rhd_i \max_{\beta\in\mathcal{E}} \left(\mathcal{M}^-_{(i,\alpha),\beta} + y_\beta\right) \text{ for all } 1 \leq i \leq k.$$

Let $\widetilde{Q} = r_1 Q_1 + \cdots + r_k Q_k$, where $Q_i = \mathrm{conv}(\mathcal{A}_i^+ \cup \mathcal{A}_i^-)$ for $i = 1, \ldots, k$, and

$$r_i = \begin{cases} \min(|\mathcal{A}_i^-|, n+1) & \text{if } \rhd_i \in \{\geq, >\} \\ \min(\max(|\mathcal{A}_i^-|, |\mathcal{A}_i^+|), n+1) & \text{if } \rhd_l \in \{=\} \end{cases} .$$

We now call **Canny-Emiris subsets** of $\mathbb{Z}^n$ associated to the pair of collections (f^+, f^-) any set $\mathcal{E}$ of the form

$$\mathcal{E} := \left(\widetilde{Q} + \delta\right) \cap \mathbb{Z}^n \quad ,$$

where δ is a generic vector in $V + \mathbb{Z}^n$, with V the direction of the affine hull of $\widetilde{Q}$.

Tropical Positivstellensatz

There exists a solution $x \in \mathbb{R}^n$ to the system $f^+(x) \rhd f^-(x)$ if and only if there exists a vector $y \in \mathbb{R}^{\mathcal{E}'}$ satisfying $\mathcal{M}^+_{\mathcal{E}'} \odot y \rhd \mathcal{M}^-_{\mathcal{E}'} \odot y$, where $\mathcal{E}'$ is any subset of $\mathbb{Z}^n$ containing a nonempty Canny-Emiris subset $\mathcal{E}$ of $\mathbb{Z}^n$ associated to the pair (f^+, f^-).

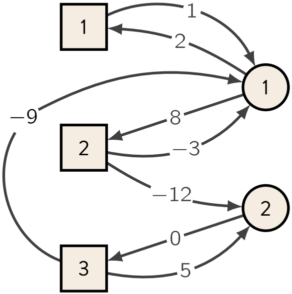
Corollary: The inclusion of basic tropical semialgebraic sets can be reduced to solving a set of tropical linear (in)equalities.

III - Mean payoff games and tropical linear systems

Mean payoff games (See **Gillette (1957)**,
Gurvich, Karzanov, Khachiyan (1988),
Zwick, Patterson (1996)):

- $G = (I \sqcup J, E)$ a (finite) oriented weighted bipartite graph;
- game with two players Min and Max: each turn, from the current state $i \in I$, player Max chooses a state $j \in J$ such that (i, j) is an arc of G with weight b_{ij} and obtains a payment of b_{ij} from player Min, then player Min from state $j \in J$, chooses the next state $k \in I$ along an arc (k, j) with weight a_{kj}, and receives in turn a payment of $a_{kj}$from player Max;
- the winner is the player who gets the highest average payment per turn;
- set $A = (a_{ij})_{(i,j)\in I\times J}$ et $B = (b_{ij})_{(i,j)\in I\times J}$.

Example : Let G be the following graph:

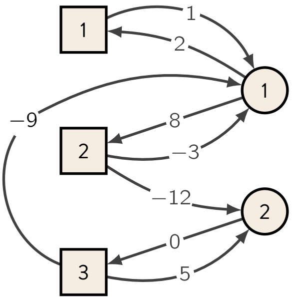


$$\text{One has } A = \begin{pmatrix} 2 & -\infty \\ 8 & -\infty \\ -\infty & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & -\infty \\ -3 & -12 \\ -9 & 5 \end{pmatrix}.$$

Theorem [Akian, Gaubert, Guterman (2012)] : For all $j \in J$, player Max has a winning positional strategy for the *mean pay-off game* given by the payment matrices A and B by playing the initial move j iff there exists a solution $y \in (\mathbb{R} \cup \{-\infty\})^J$ of the tropical matrix inequality $A \odot y \leq B \odot y$ such that $y_j \neq \mathbb{0}$.

The winning initial moves correspond to the support of the solutions of the inequality $A \odot y \leq B \odot y$.

In the previous example,



one has $\quad A \odot y \leq B \odot y \iff \begin{cases} 2 + y_1 & \leq & 1 + y_1 \\ 8 + y_1 & \leq & \max(-3 + y_1, -12 + y_2) \\ y_2 & \leq & \max(-9 + y_1, 5 + y_2). \end{cases}$

The first inequality shows that every solution $y \in (\mathbb{R} \cup \{-\infty\})^2$ must satisfy $y_1 = \mathbb{0}$, which implies that the two other inesualities are satisfied for all values of $y_2 \in \mathbb{R} \cup \{-\infty\}$.

This translates into the fact that the move 1 is a losing move for player Max, while the move 2 is a winning move.

- **Shapley operator** associated to a mean payoff game

$$T : \begin{array}{rcl} (\mathbb{R} \cup \{\pm\infty\})^J & \longrightarrow & (\mathbb{R} \cup \{\pm\infty\})^J \\ y = (y_j)_{j \in J} & \longmapsto & \left(\min_{i \in I} -a_{ij} + \left(\max_{k \in J} b_{ik} + y_k \right) \right)_{j \in J} \end{array}$$

- **value** of the game: $\chi(T) = \lim_{n \to +\infty} \frac{T^n(0)}{n}$

Corollary: $\exists y \in \mathbb{R}^J$ such that $A \odot y \leq B \odot y$ iff $\min_{j \in J} \chi_j(T) \geq 0$.

Link with nonlinear eigenvalue theory:

$$
\begin{aligned}
&\min\{\chi_j(T) : j \in J\} \\
&= \sup\{\lambda \in \mathbb{R} : \exists u \in \mathbb{R}^J, T(u) \geq \lambda + u\} \\
&= \inf\{\lambda \in \mathbb{R} \cup \{+\infty\} : \exists u \in (\mathbb{R} \cup \{+\infty\})^J, u \not\equiv +\infty, T(u) \leq \lambda + u\} \\
&= \inf\{\lambda \in \mathbb{R} \cup \{+\infty\} : \exists u \in (\mathbb{R} \cup \{+\infty\})^J, u \not\equiv +\infty, T(u) = \lambda + u\} \ .
\end{aligned}
$$

In particular $\chi(T) \equiv \lambda \in \mathbb{R}$ iff the nonlinear eigenproblem

$$T(u) = \lambda + u$$

has a solution $(\lambda, u) \in \mathbb{R} \times \mathbb{R}^J$.

The existence of a polynomial time algorithm to solve mean payoff games is an open problem since 1988, but but there exist practically fast methods (value/policy iteration algorithms).

For a Shapley operator $T : (\mathbb{R} \cup \{+\infty\})^J \to (\mathbb{R} \cup \{+\infty\})^J$, define the Krasnoselskii-Mann damped Shapley operator T_{KM} by $T_{\mathsf{KM}}(u) = \frac{u+T(u)}{2}$ for all $u \in (\mathbb{R} \cup \{+\infty\})^J$. Then $\chi(T_{\mathsf{KM}}) = \frac{\chi(T)}{2}$

We propose the following value iteration algorithm.

Value iteration algorithm

Algorithm 1: Value iteration algorithm with widening.

input: T a Shapley operator from $(\mathbb{R} \cup \{+\infty\})^J$ to $(\mathbb{R} \cup \{+\infty\})^J$
$\varepsilon > 0$ the approximation error for comparisons
N^* a timeout on the number of iterations which guarantees the existence of a solution whenever reached

output: Decides the feasibility of the system $y \leq T(y)$ in $\mathbb{R}^J$

initialization
$u := 0 \in \mathbb{R}^J$
$v := 0 \in \mathbb{R}^J$
$N := 0$
repeat
  /* Value iteration step */
  $u := v$
  $v := u \wedge T(u)$
  $N := N + 1$
  /* Widening step */
  $I := \{i : v_i \geq -\varepsilon + u_i\}$
  $\hat{u} := (\hat{u}_i) \in (\mathbb{R} \cup \{+\infty\})^m$ with $\begin{cases} \hat{u}_i = +\infty & \text{if } i \in I \\ \hat{u}_i = u_i & \text{otherwise} \end{cases}$
  $\hat{v} := T(\hat{u})$
until $v \geq -\varepsilon + u$ or $v \ll -\varepsilon + u$ or $\hat{v} \ll -\varepsilon + \hat{u}$ or $\min_{i \in J}(u_i) < -(|J| - 1)W$ or $N \geq N^*$
if $v \ll -\varepsilon + u$ or $\hat{v} \ll -\varepsilon + \hat{u}$ or $\min_{i \in J}(u_i) < -(|J| - 1)W$ **then**
  return "Unfeasible"
else
  return "Feasible"

Value iteration algorithm

Correction and termination of the value iteration algorithm

Algorithm 1 applied to the Krasnoselskii-Mann damped Shapley operator T_{KM} correctly decides (in exact arithmetic) the feasibility of a tropical linear system with integer coefficients in $N^* = \mathcal{O}(|J|^2 W)$ iterations for $\varepsilon < \frac{1}{\min(|I|,|J|)}$, where W is an upper bound on the maximal non $-\infty$ coefficients of A and B.

- Algorithm 1 is also correct and terminates in approximate arithmetics for sufficiently small approximation errors.
- Since the cost of each evaluation of the operator T is pseudo-polynomial, Algorithm 1 is in pseudo-polynomial complexity.
- To be compared with policy iteration algorithms.

Let $f^{\pm} = (f_1^{\pm}, \dots, f_k^{\pm})$ be two collections of tropical polynomials and let $d = \max_{1 \le i \le k} \deg(f_i^{\pm})$ and $W = \max_{1 \le i \le k} \|f_i^{\pm}\|_\infty$, and for $\epsilon \in \{\pm 1\}^n$, denote by $\epsilon \mathbb{R}^n_{\ge 0}$ the orthant $\{x \in \mathbb{R}^n : \epsilon_j x_j \ge 0 \text{ for all } 1 \le j \le n\}$. Then:

Short model property

- The vertices of every polyhedral complex $\{x \in \mathbb{R}^n : f_i^+(x) \ge f_i^-(x)\} \cap \epsilon \mathbb{R}^n_{\ge 0}$ are included in a $\| \cdot \|_\infty$-ball of radius $2n(2d)^{n-1}W$ centered at point 0
- Moreover, if all the coefficients of the polynomials $f_i^{\pm}$ are integer, these vertices have coordinates that are rational numbers with a denominator bounded above by $(2d)^n$.

Dichotomic search method

- Solve the system

$$\begin{cases} f^+(x) \triangleright f^-(x) \\ a \le x_1 \le b \end{cases}$$

  for varying values of a and b.
- If $|b - a| < \frac{1}{(2d)^n}$ then one can deduce the first coordinate of a solution.
- Fix the value of x_1 and repeat with $x_2, \ldots, x_n$.

The dichotomic search method returns a rational solution of this system (or decides that there is none) in $\mathcal{O}\big(\log(n(2d)^{2n-1}W)\big)$ calls to a weak mean payoff oracle.

Path-following method

- Solve the system

$$f^{+}(\zeta, x_2, \ldots, x_n) \rhd f^{-}(\zeta, x_2, \ldots, x_n)$$

for varying values of ζ.
- Linearize the above system and consider the associated mean payoff game with its shapley operator T_ζ.
- $\phi : \zeta \mapsto \min_{j \in J} \chi_j(T_\zeta)$ is a continuous, Lipschitz, piecewise affine function.
- Computing ϕ with a pivoting algorithm yields the projection of the solution set onto the first coordinate.

Python implementation of the algorithm available at:

`https://gitlab.inria.fr/abereau/tropical-polynomial-system-solving`

The bottleneck resides mainly in the computation of the Minkowski sum of the Newton polytopes of the polynomials of the system.

Open problem: Can the degree bound be improved in the Positivstellensatz (no tight example found yet)?

Thank you for your attention!

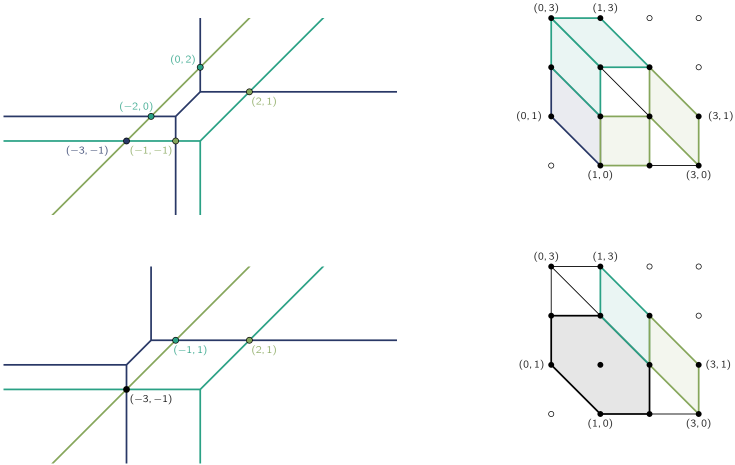
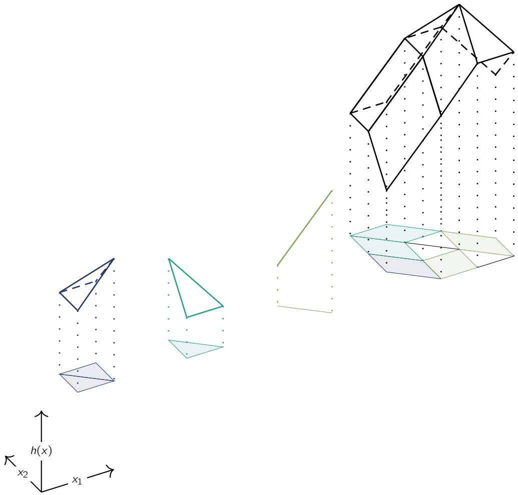
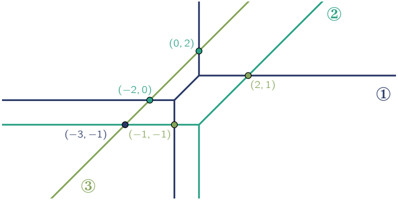


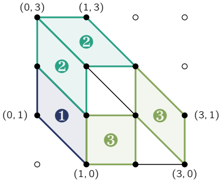
Figure: The subdivision of Q associated to (E_1) arises from the projection of the Minkowski sum of the hypographs of the lifted Newton polytopes.



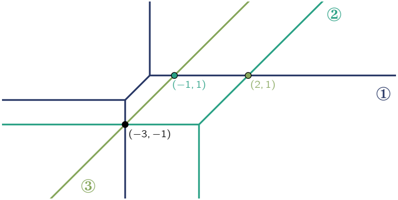
(a) The arrangement of tropical varieties of the polynomials from the system (E_1).



(b) The subdivision of Q associated to (E_1).



(c) The arrangement of tropical varieties of the polynomials from the system (E_2).



(d) The subdivision of Q associated to (E_2).

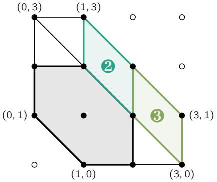
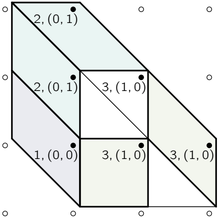


Figure: The polytope $Q + \delta$, with the integer points inside the maximal dimensional cells of the decomposition of $Q + \delta$ labelled by the row content the cell they belong to.



This configuration yields the following nonsingular square submatrix of $\mathcal{M}_{\mathcal{E}}^{(1)}$

$$\mathcal{M}_{\mathcal{E}\mathcal{E}}^{(1)} = \begin{array}{c} \\ (0,0) \to f_1 \\ (1,0) \to f_3 \\ (0,1) \to f_2 \\ (2,0) \to x_1 f_3 \\ (1,1) \to x_2 f_3 \\ (0,2) \to x_2 f_2 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & x_1 & x_2 & x_1^2 & x_1 x_2 & x_2^2 \end{array} \\ \begin{pmatrix} 1 & 2 & 1 & & 1 & \\ & 2 & 0 & & & \\ 0 & 0 & 1 & & & \\ & & & 2 & 0 & \\ & & & & 2 & 0 \\ & & 0 & & 0 & 1 \end{pmatrix} \end{array}.$$

The Shapley-Folkman Lemma

Let $A_1, \ldots, A_k \subseteq \mathbb{R}^n$, and let

$$x \in \sum_{i=1}^{k} \operatorname{conv}(A_i) \quad .$$

Then there is an index set $I \subseteq \{1, \ldots, k\}$ with $|I| \leq n$ such that

$$x \in \sum_{i \in I} \operatorname{conv}(A_i) + \sum_{i \in \{1,\ldots,k\} \setminus I} A_i \quad .$$

Corollary: If $\sum_{i=1}^{k} \operatorname{conv}(A_i)$ has (affine) dimension $d < n$, then the index set I can be choosen such that $|I| \leq d$.