The Nullstellensatz for Sparse Tropical Polynomial Systems

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- Two main concerns: find the 'smallest' suitable witness and be able to deal with sparse systems.

- **Tropical semiring** $\mathbb{R}_\infty = (\mathbb{R} \cup \{-\infty\}, \oplus, \odot)$ with
  - addition $\oplus :=$ max;
  - multiplication $\odot := +$;
  - zero element $\mathbb{0} := -\infty$;
  - unit element $\mathbb{1} := 0$.

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- Satisfies the usual properties of a field except **no additive inverse**.
- Tropical operations can be extended to vectors and matrices with coefficients in $\mathbb{R}_\infty$ allowing us to perform **tropical linear algebra**.

- A **formal tropical polynomial** p in n variables is a map

$$\begin{array}{ccc} \mathbb{Z}^n & \longrightarrow & \mathbb{R}_\infty \\ \alpha & \longmapsto & p_\alpha \end{array}$$

such that $p_\alpha \neq \mathbb{0}$ for finitely many $\alpha \in \mathbb{Z}^n$. We denote $p = \bigoplus_{\alpha \in \mathbb{Z}^n} p_\alpha X^\alpha$.

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- **Support** of p: $\operatorname{supp}(p) := \{\alpha \in \mathbb{Z}^n : p_\alpha \neq \mathbb{0}\}$
- **Polynomial function** associated to p:

$$\hat{p} : \left\{ \begin{array}{ccc} \mathbb{R}^n & \longrightarrow & \mathbb{R}_\infty \\ x & \longmapsto & \hat{p}(x) := \max_{\alpha\in\mathcal{A}}(p_\alpha + \langle x, \alpha\rangle) \end{array} \right.$$

with $\mathcal{A} = \operatorname{supp}(p)$

Example: In the system

$$(E_1):\begin{cases} f_1 &= 1\oplus 2x_1\oplus 1x_2\oplus 1x_1x_2\\ f_2 &= 0\oplus 0x_1\oplus 1x_2\\ f_3 &= 2x_1\oplus 0x_2\ ,\end{cases}$$

the polynomials have respective supports

$$\begin{cases} \mathcal{A}_1 &= \{(0,0),(1,0),(0,1),(1,1)\}\\ \mathcal{A}_2 &= \{(0,0),(1,0),(0,1)\}\\ \mathcal{A}_3 &= \{(1,0),(0,1)\}\ .\end{cases}$$

Same goes for the system

$$(E_2):\begin{cases} f_1 &= 1\oplus 4x_1\oplus 1x_2\oplus 3x_1x_2\\ f_2 &= 0\oplus 0x_1\oplus 1x_2\\ f_3 &= 2x_1\oplus 0x_2\ ,\end{cases}$$

- $x \in \mathbb{R}_\infty^n$ is a **root** of a polynomial p whenever the maximum in the expression

$$\hat{p}(x) = \bigoplus_{\alpha \in \mathcal{A}} p_\alpha \odot x^{\odot \alpha} = \max_{\alpha \in \mathcal{A}} (p_\alpha + \langle x, \alpha \rangle)$$

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is attained twice. This is denoted as $p(x) \nabla \mathbb{0}$.

- $y \in \mathbb{R}^m$ is in the **tropical right null space** of a $\ell \times m$ matrix $A = (a_{ij})$ whenever for all $1 \leq i \leq \ell$, the maximum in the expression

$$\bigoplus_{j=1}^{m} a_{ij} \odot y_j = \max_{1 \leq j \leq m} (a_{ij} + y_j)$$

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Figure: The arrangement of tropical varieties of the polynomials from the system

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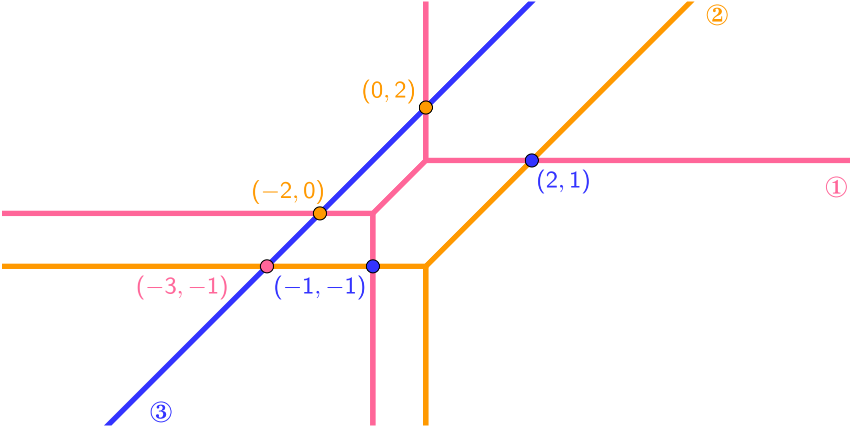

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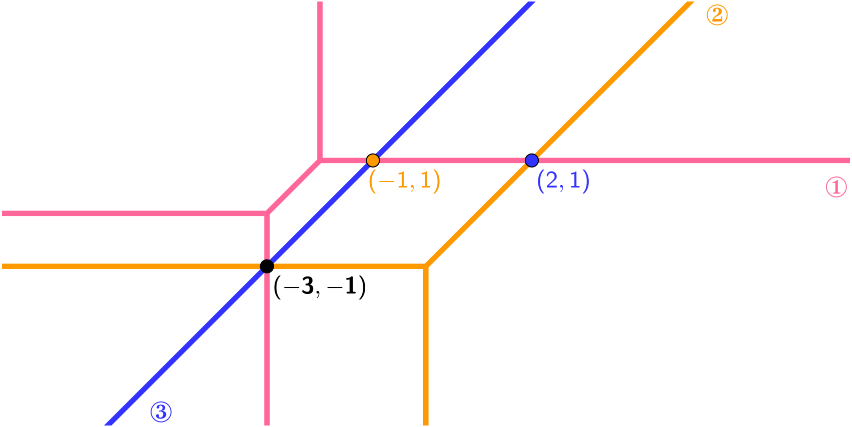

In the following, we fix a collection $f = (f_1, \ldots, f_k)$ of k formal tropical polynomials in n variables, with respective supports $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_k)$ and degrees $(d_1, \ldots, d_k)$.

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Remark: The same question exists for solution in $\mathbb{R}^n_\infty$. It reduces to the $\mathbb{R}^n$ by considering the supports.

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$$\mathcal{W}_{\mathbb{K}}(\mathcal{A}) = \left\{(f, x) \in \left((\mathbb{K}^*)^{\mathcal{A}_1} \times \cdots \times (\mathbb{K}^*)^{\mathcal{A}_k}\right) \times (\mathbb{K}^*)^n : \forall i, f_i(x) = 0\right\}.$$

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Now consider $\mathcal{Z}_{\mathbb{K}}(\mathcal{A})$ the projection of $\mathcal{W}_{\mathbb{K}}(\mathcal{A})$ onto the first factor of the cartesian product, that is

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Then, we define the **resultant variety** $\mathcal{RV}_{\mathbb{K}}(\mathcal{A})$ associated to $\mathcal{A}$ as the Zariski closure $\overline{\mathcal{Z}_{\mathbb{K}}(\mathcal{A})}$ of the previous set.

Similarly, the **tropical incidence variety** associated to $\mathcal{A}$ is defined as the set

$$\mathcal{W}_{\text{trop}}(\mathcal{A}) = \left\{(f, x) \in \left(\mathbb{R}^{\mathcal{A}_1} \times \cdots \times \mathbb{R}^{\mathcal{A}_k}\right) \times \mathbb{R}^n : \forall i, f_i(x) \ \nabla \ \mathbb{0}\right\},$$

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and the projection of $\mathcal{W}_{\text{trop}}(\mathcal{A})$ on the first factor is the set

$$\mathcal{Z}_{\text{trop}}(\mathcal{A}) = \left\{f \in \mathbb{R}^{\mathcal{A}_1} \times \cdots \times \mathbb{R}^{\mathcal{A}_k} : \exists x \in \mathbb{R}^n,\ \forall i, f_i(x) \nabla \mathbb{0}\right\},$$

but this time the **tropical resultant variety** $\mathcal{RV}_{\text{trop}}(\mathcal{A})$ associated to $\mathcal{A}$ is simply defined by $\mathcal{RV}_{\text{trop}}(\mathcal{A}) = \mathcal{Z}_{\text{trop}}(\mathcal{A})$.

If $k = n + 1$ and if the resultant variety $\mathcal{RV}_{\mathbb{K}}(\mathcal{A})$ is a hypersurface, *i.e.* that it is described by a single irreducible polynomial equation of the form $\mathcal{R}(f) = 0$, then we call the polynomial $\mathcal{R}$ the **sparse resultant polynomial** of the family f.

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Moreover, if $\mathcal{RV}_{\mathbb{K}}(\mathcal{A})$ is not a hypersurface, then the resultant is simply taken to be the constant polynomial equal to 1.

The fact that $\mathcal{RV}_{\mathbb{K}}(\mathcal{A})$ is an hypersurface can be characterized by a combinatorial condition on the collection of supports $\mathcal{A}$.

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In other words,

$$\mathcal{R}_{\text{trop}}(f) = \bigoplus_{\nu \in M} f^{\nu} \ ,$$

where M is the set of exponents of essential monomials of the classical resultant polynomial.

The theory of tropical resultant varieties was explored by Jensen and Yu (2013) who showed that the tropical resultant variety coincides with the tropicalisation of the classical resultant variety, following the work of Sturmfels (1994) and coauthors who characterized the Newton polytope of the classical resultant.

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From these results, we were able to show the following theorem:

Tropical resultant

Under the previous assumptions for $k = n + 1$, the following assertions are equivalent:

1. There exists $x \in \mathbb{R}^n$ such that $f_i(x) \, \nabla \, \mathbb{0}$ for all $1 \leq i \leq n + 1$;
2. $\mathcal{R}_{\text{trop}}(f) \, \nabla \, \mathbb{0}$;

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- A finite subset $\mathcal{E}$ of $\mathbb{Z}^n$ yields a (finite) submatrix $\mathcal{M}_\mathcal{E}$ of $\mathcal{M}$ obtained by taking only the rows whose support is included in $\mathcal{E}$ and the columns indexed by $\mathcal{E}$.

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- For $\mathcal{E} = \{\alpha \in \mathbb{N}^n : \alpha_1 + \cdots + \alpha_n \leq N\}$, we denote $\mathcal{M}_N := \mathcal{M}_\mathcal{E}$.

A tropical Nullstellensatz was established by Grigoriev and Podolskii (2018) for full polynomials. It uses the submatrix $\mathcal{M}_N$ of the Macaulay matrix $\mathcal{M}$ obtained by truncating it to the degree $N = (n+2)(d_1 + \cdots + d_k)$.

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Tropical Dual Nullstellensatz [Grigoriev and Podolskii (2018)]

The polynomials of f have a common root $x \in \mathbb{R}^n$ iff there exists a vector $y \in \mathbb{R}^m$ with $m = \binom{N+n}{n}$ in the tropical right null space of the truncated Macaulay matrix $\mathcal{M}_N$ for

$$N = (n+2)(d_1 + \cdots + d_k) \; .$$

- The problem of finding a vector $y \in \mathbb{R}^n$ such that $\mathcal{M}_N \odot y \, \nabla \, \mathbb{0}$ can be solved using mean payoff games cf Akian, Gaubert and Guterman (2009).

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- However, the truncation degree of Grigoriev and Podolskii does not match the Macaulay degree in the classical theory which is equal to $d_1 + \cdots + d_{n+1} - n$ in the case $k = n + 1$. Their proof did not seem to allow room for improving this bound.

This work improves on Grigoriev and Podolskii's result by taking into account the sparse structure of the polynomials, and connects the tropical Nullstellensatz with classical elimination theory. In particular, it relies on a construction by Canny and Emiris (1993) and Strumfels (1994).

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This result in an improved truncation degree and allows us to deal better with sparse polynomials.

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- If $h := h_1 \,\Box \cdots \Box\, h_k$ where $\Box$ denotes the sup-convolution, then $\text{hypo}(h) = \text{hypo}(h_1) + \cdots + \text{hypo}(h_k)$ and moreover the supports of h is $Q = Q_1 + \cdots + Q_k$.

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- The projection of $\mathrm{hypo}(h)$ onto Q yields a **coherent mixed subdivision** of Q.

The Newton polytopes associated to both systems (E_1) and (E_2) and their Minkowski sum are as follow.

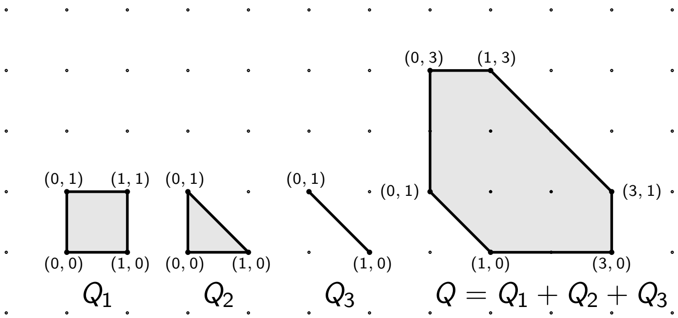
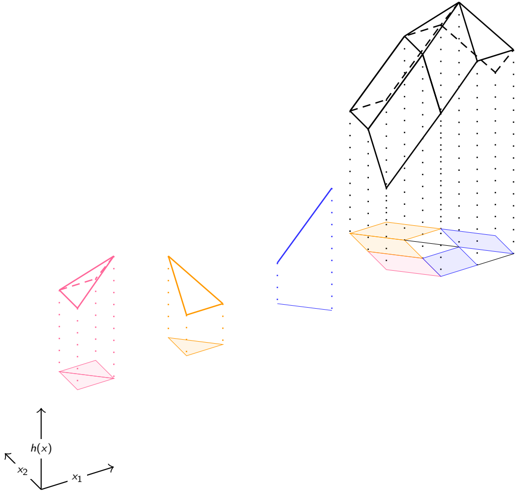
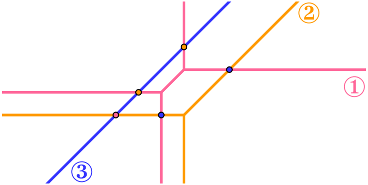


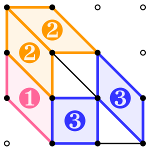
Figure: The subdivision of Q associated to (E_1) arises from the projection of the Minkowski sum of the hypographs of the h_i.



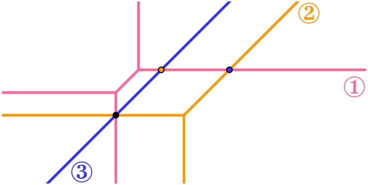
(a) The arrangement of tropical varieties of the polynomials from the system (E_1).



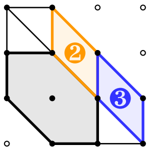
(b) The subdivision of Q associated to (E_1).



(c) The arrangement of tropical varieties of the polynomials from the system (E_2).



(d) The subdivision of Q associated to (E_2).



Canny-Emiris set associated to f: $\mathcal{E} = (Q + \delta) \cap \mathbb{Z}^n$ with δ a generic vector in the linear space directing the affine hull of Q.

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Example: Considering again the systems (E_1) and (E_2), for

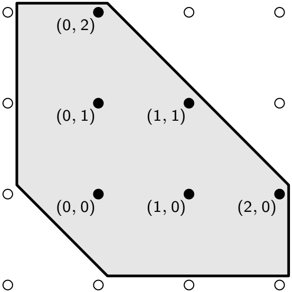
$$\delta = (-1 + \varepsilon, -1 + \varepsilon)$$

with $\varepsilon > 0$ sufficiently small, we obtain the Canny-Emiris set

$$\mathcal{E} := (Q + \delta) \cap \mathbb{Z}^n = \{(0,0), (1,0), (0,1), (2,0), (1,1), (0,2)\}$$

corresponding to the set of monomials $\{1, x_1, x_2, x_1^2, x_1 x_2, x_2^2\}$.

Figure: The polytope $Q + \delta$ with $\delta = (-0.9, -0.9)$.



Nullstellensatz for Sparse Tropical Polynomial Systems

The system $f \nabla \mathbb{0}$ has a solution $x \in \mathbb{R}^n$ iff there exists a vector $y \in \mathbb{R}^{\mathcal{E}'}$ in the tropical right null space of the submatrix $\mathcal{M}_{\mathcal{E}'}$ of $\mathcal{M}$, where $\mathcal{E}'$ is any subset of $\mathbb{Z}^n$ containing a nonempty Canny-Emiris set $\mathcal{E}$.

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Corollary: *The system $f \nabla \mathbb{0}$ has a solution $x \in \mathbb{R}^n$ if and only if the truncated Macaulay tropical linear system $\mathcal{M}_N \odot y \nabla \mathbb{0}$ has a solution $y \in \mathbb{R}^m$ for*

$$N = d_1 + \cdots + d_k \quad ,$$

where $d_i = \deg(f_i)$ for all $1 \leq i \leq k$.

Example: The matrix associated with system (E_1) is

$$\mathcal{M}_{\mathcal{E}}^{(1)} = \begin{array}{r} \\ f_1 \\ f_2 \\ x_1 f_2 \\ x_2 f_2 \\ f_3 \\ x_1 f_3 \\ x_2 f_3 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & x_1 & x_2 & x_1^2 & x_1 x_2 & x_2^2 \end{array} \\ \begin{pmatrix} 1 & 2 & 1 & & 1 & \\ 0 & 0 & 1 & & & \\ & 0 & & 0 & 1 & \\ & & 0 & & 0 & 1 \\ & 2 & 0 & & & \\ & & & 2 & 0 & \\ & & & & 2 & 0 \end{pmatrix} \end{array} .$$

There is no finite vector in its tropical right null space and thus there is no finite solution to the equation $f \nabla \mathbb{0}$.

Example: The matrix associated with system (E_2) is

$$\mathcal{M}_{\mathcal{E}}^{(2)} = \begin{array}{r} \\ f_1 \\ f_2 \\ x_1 f_2 \\ x_2 f_2 \\ f_3 \\ x_1 f_3 \\ x_2 f_3 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & x_1 & x_2 & x_1^2 & x_1 x_2 & x_2^2 \end{array} \\ \begin{pmatrix} 1 & 4 & 1 & & 3 & \\ 0 & 0 & 1 & & & \\ & 0 & & 0 & 1 & \\ & & 0 & & 0 & 1 \\ & 2 & 0 & & & \\ & & & 2 & 0 & \\ & & & & 2 & 0 \end{pmatrix} \end{array} .$$

The vector $y = \mathrm{ver}(-3,-1) = (0,-3,-1,-6,-4,-2)$ is finite and is in the tropical right null space of the previous matrix, hence there is a finite solution to the equation $f \nabla \mathbb{0}$, which is indeed given by $(-3,-1)$.

A $d \times d$ tropical matrix $A = (a_{ij})_{1 \leq i,j \leq d}$ is **tropically diagonally dominant** whenever

$$a_{ii} > a_{ij}$$

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Lemma: *If A is tropically diagonally dominant, then the only solution $y \in \mathbb{R}^d_\infty$ to the equation $A \odot y \ \nabla\ \mathbb{0}$ is $y = \mathbb{0}$.*

Proof: Consider $y_i = \max_{1 \leq j \leq n} y_j$, then if $y_i > -\infty$ then the inequalities $a_{ii} > a_{ij}$ and $y_i \geq y_j$ imply that

$$a_{ii} + y_i > a_{ij} + y_j \quad \text{for all} \quad 1 \leq i \neq j \leq n \ ,$$

thus contradicting the assumption that $A \odot y \ \nabla\ \mathbb{0}$.

- If $f \nabla \mathbb{0}$ has a solution $x \in \mathbb{R}^n$, then the **Veronese embedding** $y = \mathrm{ver}(x) := (x^p)_{p \in \mathcal{E}'}$ of x is a solution to $\mathcal{M}_{\mathcal{E}'} \odot y \nabla \mathbb{0}$.

- If $f \nabla \mathbb{0}$ has a solution $x \in \mathbb{R}^n$, then the **Veronese embedding** $y = \mathrm{ver}(x) := (x^p)_{p \in \mathcal{E}'}$ of x is a solution to $\mathcal{M}_{\mathcal{E}'} \odot y \nabla \mathbb{0}$.
- Otherwise we apply a construction from Canny and Emiris (1993) and Sturmfels (1994) but in a potentially **non generic** case to show that there is no finite vector $y \in \mathbb{R}^{\mathcal{E}'}$ in the tropical right null space of $\mathcal{M}_{\mathcal{E}'}$.

- If $p \in \mathcal{E}$, then $(p-\delta, h(p-\delta))$ is in the **relative interior** of a facet F of hypo(h), and F can be written as $F_1 + \cdots + F_k$ with F_i faces of hypo(h_i).

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- Since f does not have a common root, at least one F_i is a singleton. Consider the maximal index j such that $F_j = \{a_j\}$ is a **singleton**. The couple (j, a_j) is called the **row content** of p.

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- If $p \in \mathcal{E}$ and if (j, a_j) is its row content, then the support of the polynomial $X^{p-a_j} f_j$ is included in $\mathcal{E}$. This allows us to construct a square submatrix $\mathcal{M}_{\mathcal{E}\mathcal{E}} = (m_{pp'})_{(p,p') \in \mathcal{E} \times \mathcal{E}}$ of $\mathcal{M}_{\mathcal{E}}$.

- The matrix $\widetilde{\mathcal{M}}_{\mathcal{E}\mathcal{E}} = (\widetilde{m}_{pp'})_{(p,p')\in\mathcal{E}\times\mathcal{E}}$ obtained by setting $\widetilde{m}_{pp'} = m_{pp'} - h(p' - \delta)$ is **tropically diagonally dominant**.

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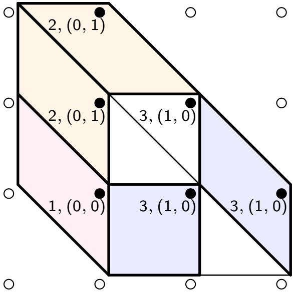
- Therefore its tropical right null space is reduced to $\{\mathbb{0}\}$, and thus this is also the case for $\mathcal{M}_{\mathcal{E}\mathcal{E}}$.
- Hence there does not exist $y \in \mathbb{R}^{\mathcal{E}}$ such that $\mathcal{M}_{\mathcal{E}} \odot y \,\nabla\, \mathbb{0}$.
- Finally, since $\mathcal{M}_{\mathcal{E}'}$ can be written by block as

$$\mathcal{M}_{\mathcal{E}'} = \begin{array}{c} \begin{matrix} \mathcal{E} & \quad \mathcal{E}'\setminus\mathcal{E} \end{matrix} \\ \begin{pmatrix} \mathcal{M}_{\mathcal{E}} & \mathbb{0} \\ * & * \end{pmatrix} \end{array},$$

we deduce that there does also not exist $y \in \mathbb{R}^{\mathcal{E}'}$ such that $\mathcal{M}_{\mathcal{E}'} \odot y \,\nabla\, \mathbb{0}$.

Figure: The polytope $Q + \delta$, with interior integer points labelled by the row content of the cell they belong to for system (E_1).



For the system (E_1), we obtain the matrix

$$\mathcal{M}_{\mathcal{E}\mathcal{E}} = \begin{array}{ccc} (0,0) & \to & f_1 \\ (1,0) & \to & f_3 \\ (0,1) & \to & f_2 \\ (2,0) & \to & x_1 f_3 \\ (1,1) & \to & x_2 f_3 \\ (0,2) & \to & x_2 f_2 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 \end{array} \\ \begin{pmatrix} 1 & 2 & 1 & & 1 & \\ & 2 & 0 & & & \\ 0 & 0 & 1 & & & \\ & & & 2 & 0 & \\ & & & & 2 & 0 \\ & & 0 & & 0 & 1 \end{pmatrix} \end{array}$$

which is nonsingular.

Proof of the corollary: Take $\mathcal{E}'$ to be the simplex

$$\{\alpha \in \mathbb{N}^n : \alpha_1 + \cdots + \alpha_k \leq N\} \ .$$

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Remarks:

- This value of $N = d_1 + \cdots + d_k$ improves on Grigoriev and Podolskii's result (2018) which gave

$$N = (n+2)(d_1 + \cdots + d_k) \ .$$

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$$N = (n+2)(d_1 + \cdots + d_k) \ .$$
Moreover, the general theorem also allows to tackle the sparse case.
- In most cases, we can even obtain the bound $N = d_1 + \cdots + d_k - (k-1)$ which corresponds to the bound in the usual polynomial case.

Bipartite systems $f^+ \geq f^-$ or $f^+ = f^-$ with $f^\pm = (f_1^\pm, \dots, f_k^\pm)$, *i.e.*

$$f_i^+(x) \geq f_i^-(x) \quad \text{for all} \quad 1 \leq i \leq k$$

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For $f^+ \geq f^-$, if f^- has a 'small' number of monomials, then the inequality can be reduced to the case where all polynomials f_i^- are monomials.

In the case where all polynomials f_i^- are monomials, we consider $f = f^+ \oplus f^-$ and we apply the previous construction to f in order to get a Canny-Emiris set $\mathcal{E}$.

In the case where all polynomials f_i^- are monomials, we consider $f = f^+ \oplus f^-$ and we apply the previous construction to f in order to get a Canny-Emiris set $\mathcal{E}$. We then have the following result:

Positivstellensatz for Sparse Systems of Tropical Polynomials

There exists a solution $x \in \mathbb{R}^n$ to the inequation $f^+ \geq f^-$ if and only if there exists a vector $y \in \mathbb{R}^{\mathcal{E}'}$ satisfying the inequality $\mathcal{M}^+_{\mathcal{E}'} \odot y \geq \mathcal{M}^-_{\mathcal{E}'} \odot y$, where $\mathcal{E}'$ is any subset of $\mathbb{Z}^n$ containing a nonempty Canny-Emiris subset $\mathcal{E}$ of $\mathbb{Z}^n$ associated to the system $f^+ \geq f^-$.

In the case where there are too many monomials, we apply the Canny-Emiris construction to $(n+1)Q$ instead of Q and use the Shapley-Folkman theorem in order to define the notion of row content in this case.

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- Unify these two cases;

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- Tackle the case $f^+ = f^-$;

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- Unify these two cases;
- Tackle the case $f^+ = f^-$;
- Develop eigenvalue methods to solve effectively tropical polynomial systems.

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Thank you for your attention!