The Nullstellensatz for Sparse Tropical Polynomial Systems

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We fix a collection $f = (f_1, \dots, f_k)$ of tropical polynomials in n variables with respective degrees $(d_1, \dots, d_k)$.

Motivation

Theorem (Tropical Dual Nullstellensatz, [GP18]) *The polynomials of f have a common root $x \in \mathbb{R}^n$ iff there exists a vector $y \in \mathbb{R}^m$ with $m = \binom{N+n}{n}$ in the tropical right null space of the truncated Macaulay matrix $\mathcal{M}_N$ for*

$$N = (n+2)(d_1 + \cdots + d_k) \ .$$

Question : What is the smallest possible value of N such that this result holds and how can this result be expressed for **sparse** polynomials?

Tropical Polynomials

- **Tropical semiring**: $\mathbb{R}_\infty = (\mathbb{R} \cup \{-\infty\}, \oplus, \odot)$ with addition $\oplus := \max$, multiplication $\odot := +$, zero element $\mathbb{0} := -\infty$ and unit element $\mathbb{1} := 0$.
- $x \in \mathbb{R}^n_\infty$ is a **root** of a tropical polynomial p whenever the maximum in the expression
$$p(x) = \bigoplus_{\alpha \in \mathcal{A}} p_\alpha \odot x^{\odot \alpha} = \max_{\alpha \in \mathcal{A}} (p_\alpha + \langle x, \alpha \rangle)$$
is attained twice, where $\mathcal{A} \subset \mathbb{Z}^n$ is the support of p. This is denoted as $p(x) \ \nabla \ \mathbb{0}$.
- y is in the **tropical right null space** of a $\ell \times m$ matrix $A = (a_{ij})$ whenever for all $1 \leq i \leq \ell$, the maximum in the expression
$$\bigoplus_{j=1}^{m} a_{ij} \odot y_j = \max_{1 \leq j \leq m} (a_{ij} + y_j)$$
is attained twice. This is denoted as $A \odot y \ \nabla \ \mathbb{0}$.

The Macaulay matrix

- The **Macaulay matrix** associated to f is the matrix $\mathcal{M} = (m_{(i,\alpha),\beta})$ indexed by $([n] \times \mathbb{Z}^n) \times \mathbb{Z}^n$, where $m_{(i,\alpha),\beta}$ corresponds to the coefficient of X^β in the tropical polynomial $X^\alpha f_i$.
- A finite subset $\mathcal{E}$ of $\mathbb{Z}^n$ yields a submatrix $\mathcal{M}_\mathcal{E}$ of $\mathcal{M}$ obtained by taking only the rows whose support is included in $\mathcal{E}$ and the columns indexed by $\mathcal{E}$.
- For $\mathcal{E} = \{\alpha \in \mathbb{N}^n : \alpha_1 + \cdots \alpha_n \leq N\}$, we denote $\mathcal{M}_N := \mathcal{M}_\mathcal{E}$.

Newton polytopes

- Set for $1 \leq i \leq k$, $Q_i := \mathrm{conv}(\mathcal{A}_i)$ the **Newton polytope** of f_i and $Q = Q_1 + \cdots + Q_k$.
- The upper hull of the lifted support $\{(\alpha, f_{i,\alpha}) : \alpha \in \mathcal{A}_i\}$ is the graph of a function h_i with support Q_i and if $h := h_1 \square \cdots \square h_k$ where $\square$ denotes the sup-convolution, then $\mathrm{hypo}(h) = \mathrm{hypo}(h_1) + \cdots + \mathrm{hypo}(h_k)$.
- The projection of $\mathrm{hypo}(h)$ onto Q yields a coherent mixed subdivision of Q.

Canny-Emiris subsets of $\mathbb{Z}^n$

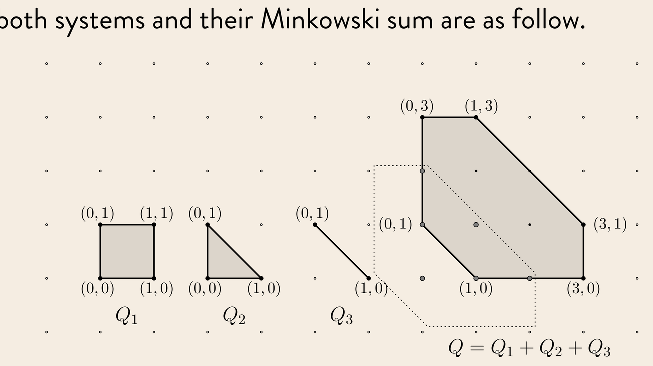
Canny-Emiris set associated to f: any set of the form $\mathcal{E} = (Q + \delta) \cap \mathbb{Z}^n$ with δ a generic vector in the linear space directing the affine hull of Q.

Examples

Consider

$$(E_1) : \begin{cases} f_1 = 1 \oplus 2x_1 \oplus 1x_2 \oplus 1x_1x_2 \\ f_2 = 0 \oplus 0x_1 \oplus 1x_2 \\ f_3 = 2x_1 \oplus 0x_2 \end{cases} \quad \text{and} \quad (E_2) : \begin{cases} f_1 = 1 \oplus 4x_1 \oplus 1x_2 \oplus 3x_1x_2 \\ f_2 = 0 \oplus 0x_1 \oplus 1x_2 \\ f_3 = 2x_1 \oplus 0x_2 \ . \end{cases}$$

The Newton polytopes associated to both systems and their Minkowski sum are as follow.



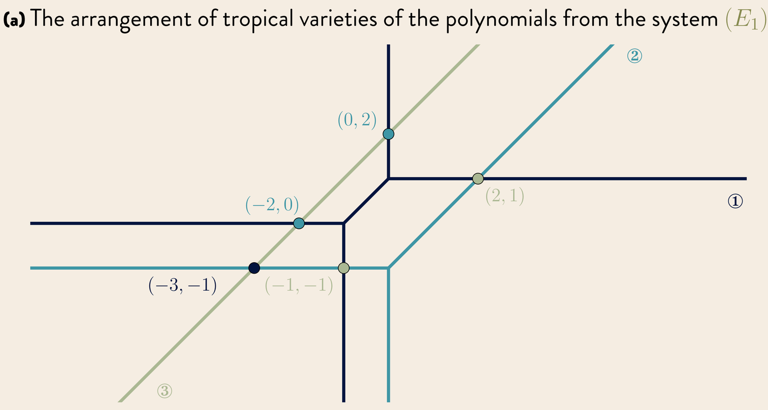
For $\delta = (-1 + \varepsilon, -1 + \varepsilon)$ with $\varepsilon > 0$ sufficiently small, we obtain the Canny-Emiris set

$$\mathcal{E} := (Q + \delta) \cap \mathbb{Z}^n = \{(0,0), (1,0), (0,1), (2,0), (1,1), (0,2)\}$$

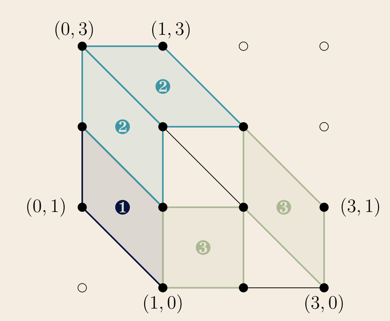
corresponding to the set of monomials $\{1, x_1, x_2, x_1^2, x_1x_2, x_2^2\}$. The associated matrices are

$$\mathcal{M}^{(1)}_\mathcal{E} = \begin{array}{c} \\ f_1 \\ f_2 \\ x_1f_2 \\ x_2f_2 \\ f_3 \\ x_1f_3 \\ x_2f_3 \end{array} \begin{array}{c} \begin{matrix} 1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 \end{matrix} \\ \begin{pmatrix} 1 & 2 & 1 & & 1 & \\ 0 & 0 & 1 & & & \\ & 0 & & 0 & 1 & \\ & & 0 & & 0 & 1 \\ & 2 & 0 & & & \\ & & & 2 & 0 & \\ & & & & 2 & 0 \end{pmatrix} \end{array} \quad \text{and} \quad \mathcal{M}^{(2)}_\mathcal{E} = \begin{array}{c} \\ f_1 \\ f_2 \\ x_1f_2 \\ x_2f_2 \\ f_3 \\ x_1f_3 \\ x_2f_3 \end{array} \begin{array}{c} \begin{matrix} 1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 \end{matrix} \\ \begin{pmatrix} 1 & 4 & 1 & & 3 & \\ 0 & 0 & 1 & & & \\ & 0 & & 0 & 1 & \\ & & 0 & & 0 & 1 \\ & 2 & 0 & & & \\ & & & 2 & 0 & \\ & & & & 2 & 0 \end{pmatrix} \end{array} .$$

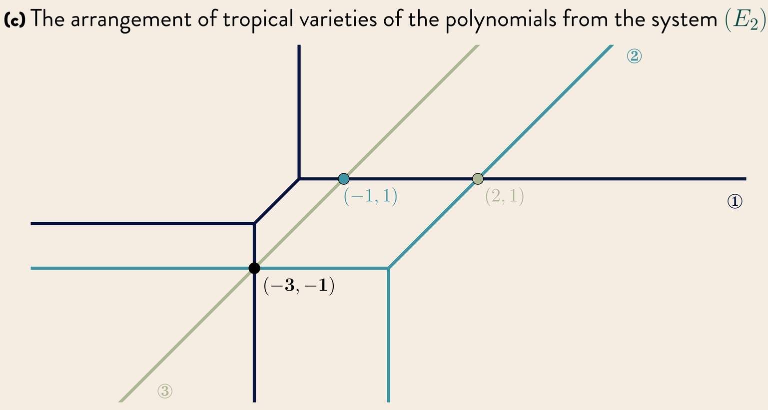
(a) The arrangement of tropical varieties of the polynomials from the system (E_1).



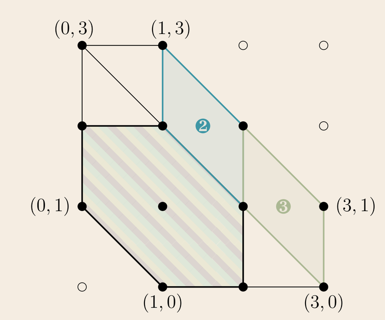
(b) The subdivision of Q associated to (E_1).



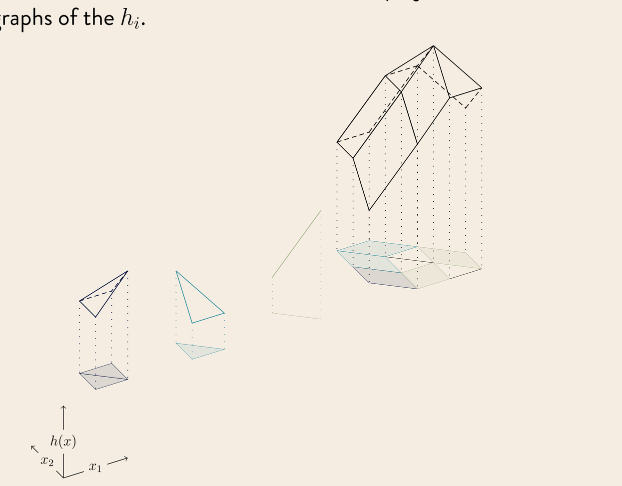
(c) The arrangement of tropical varieties of the polynomials from the system (E_2).



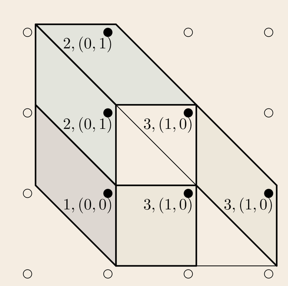
(d) The subdivision of Q associated to (E_2).



(e) The subdivision of Q associated to (E_1) arises from the projection of the Minkowski sum of the hypographs of the h_i.



(f) The polytope $Q + \delta$, with interior integer points labelled by the row content of the cell they belong to for system (E_1).



For the system (E_1), we obtain the matrix

$$\mathcal{M}^{(1)}_{\mathcal{E}\mathcal{E}} = \begin{array}{c} \\ (0,0) \to f_1 \\ (1,0) \to f_3 \\ (0,1) \to f_2 \\ (2,0) \to x_1f_3 \\ (1,1) \to x_2f_3 \\ (0,2) \to x_2f_2 \end{array} \begin{array}{c} \begin{matrix} 1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 \end{matrix} \\ \begin{pmatrix} 1 & 2 & 1 & & 1 & \\ & 2 & 0 & & & \\ 0 & 0 & 1 & & & \\ & & & 2 & 0 & \\ & & & & 2 & 0 \\ & & 0 & & 0 & 1 \end{pmatrix} \end{array}$$

which is nonsingular.

For the system (E_2), for $y = \mathrm{ver}(-3, -1) = (0, -3, -1, -6, -4, -2)$, we obtain $\mathcal{M}^{(2)}_\mathcal{E} \odot y \ \nabla \ \mathbb{0}$.

Nullstellensatz for Sparse Tropical Polynomial Systems

Theorem *The system $f \ \nabla \ \mathbb{0}$ has a solution $x \in \mathbb{R}^n$ iff there exists a vector $y \in \mathbb{R}^{\mathcal{E}'}$ in the tropical right null space of the submatrix $\mathcal{M}_{\mathcal{E}'}$ of $\mathcal{M}$, where $\mathcal{E}'$ is any subset of $\mathbb{Z}^n$ containing a nonempty Canny-Emiris set $\mathcal{E}$.*

Corollary *The system $f \ \nabla \ \mathbb{0}$ has a solution $x \in \mathbb{R}^n$ if and only if the truncated Macaulay tropical linear system $\mathcal{M}_N \odot y \ \nabla \ \mathbb{0}$ has a solution $y \in \mathbb{R}^m$ for $N = d_1 + \cdots + d_k$.*

The Canny-Emiris construction

- If $f \ \nabla \ \mathbb{0}$ has a solution $x \in \mathbb{R}^n$, then $y = (x_p)_{p \in \mathcal{E}'}$ of x is a solution to $\mathcal{M}_{\mathcal{E}'} \odot y \ \nabla \ \mathbb{0}$.
- Otherwise we apply the Canny-Emiris construction from [CE93] and [Stu94] but in a potentially **non generic** case. If $p \in Q$, then $(p - \delta, h(p - \delta))$ is in the relative interior of a facet F of $\mathrm{hypo}(h)$, and F can be written as $F_1 + \cdots + F_k$ with F_i faces of $\mathrm{hypo}(h_i)$.
- Since f does not have a common root, at least one F_i is a singleton. Consider the maximal index j such that $F_j = \{a_j\}$ is a singleton. The couple (j, a_j) is called the **row content** of p.
- If $p \in \mathcal{E}$ and if (j, a_j) is its row content, then the support of the polynomial $X^{p - a_j} f_j$ is included in $\mathcal{E}$. This allows us to construct a square submatrix $\mathcal{M}_{\mathcal{E}\mathcal{E}} = (m_{pp'})_{(p,p') \in \mathcal{E} \times \mathcal{E}}$ of $\mathcal{M}_\mathcal{E}$.
- The matrix $\widetilde{\mathcal{M}}_{\mathcal{E}\mathcal{E}} = (\widetilde{m}_{pp'})_{(p,p') \in \mathcal{E} \times \mathcal{E}}$ obtained by setting $\widetilde{m}_{pp'} = m_{pp'} - h(p' - \delta)$ is **tropically diagonally dominant**, and therefore its tropical right null space is reduced to $\{\mathbb{0}\}$, and thus this is also the case for $\mathcal{M}_{\mathcal{E}\mathcal{E}}$, hence there does not exist $y \in \mathbb{R}^\mathcal{E}$ such that $\mathcal{M}_\mathcal{E} \odot y \ \nabla \ \mathbb{0}$.

Perspectives and related results

- We can in fact retrieve the Macaulay bound in most cases.
- **Tropical resultant polynomial** based on [JY13] and generalizing tropical Cramer's theorem from [AGG08] in the case $k = n + 1$ but no tropical determinantal formula yet.
- Work in progress: Nullstellensatz for **two-sided systems** of the form $f^+ \geq f^-$, $f^+ = f^-$ and $f^+ > f^-$, relying on the Shapley-Folkman and adding a factor $n + 1$ in the truncation degree N:

Positivstellensatz for Sparse Tropical Polynomial Systems

Theorem *For $\diamond \in \{\geq, =, >\}$, the polynomial system $f^+ \diamond f^-$ has a solution $x \in \mathbb{R}^n$ iff the linear system $\mathcal{M}^+_{\mathcal{E}'} \diamond \mathcal{M}^-_{\mathcal{E}'}$ has a solution $y \in \mathbb{R}^{\mathcal{E}'}$, where $\mathcal{E}'$ is any subset of $\mathbb{Z}^n$ containing a nonempty Canny-Emiris set $\mathcal{E} = ((\boldsymbol{n+1})Q + \delta) \cap \mathbb{Z}^n$.*

Corollary *We can similarly deal with mixed systems, and in particular with problems of the form*

$$f_1^+ \geq f_1^-, \dots, f_k^+ \geq f_k^- \overset{?}{\Rightarrow} g^+ \geq g^- \ .$$

- Incoming work: tropical eigenvalue method to solve effectively tropical polynomial systems.

References

[AGG08] Marianne Akian, Stéphane Gaubert and Alexander Guterman, Linear independence over tropical semirings and beyond, *Contemp. Math.* (2008) **495**

[CE93] John Canny and Ioannis Emiris, An efficient algorithm for the sparse mixed resultant (1993) **263**:89-104

[GP18] Dima Grigoriev and Vladimir V. Podolskii, Tropical Effective Primary and Dual Nullstellensätze, *Discrete Comput Geom* (2018) **59**:507-522

[JY13] Anders Jensen and Josephine Yu, Computing tropical resultants, *Journal of Algebra* (2013) **387**:287-319.

[Stu94] Bernd Sturmfels, On the Newton Polytope of the Resultant, *Journal of Algebraic Combinatorics* (1994) **3**(2):207-236